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The NP-Completeness Column: An Ongoing Guide

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This is the second edition of a quarterly column the purpose of which is to provide a continuing update to the list of problems (NP-complete and harder) presented by M. R. Garey and myself in our book ''Computers and Intractability: A Guide to the Theory of NP-Completeness,'' W. H. Freeman & Co., San Francisco, 1979 (hereinafter referred to as ''[G&J]''; previous columns will be referred to by their dates). A background equivalent to that provided by [G&J] is assumed. Readers having results they would like mentioned (NP-hardness, PSPACE-hardness, polynomial-time-solvability, etc.), or open problems they would like publicized, should send them to David S. Johnson, Room 2C-355, Bell Laboratories, Murray Hill, NJ 07974, including details, or at least sketches, of any new proofs (full papers are preferred). In the case of unpublished results, please state explicitly that you would like the results mentioned in the column. Comments and corrections are also welcome. For more details on the nature of the column and the form of desired submissions, see the December 1981 issue of this journal.

EMBEDDING PROBLEMS

Many combinatorial problems ask about embeddings of graphs into other objects. For instance, the polynomial time solvable GRAPH PLANARITY problem asks whether a given graph G can be embedded in the plane in such a way that no two edges intersect (except at a common endpoint). The open GRAPH GENUS problem [Dec. 1981] asks whether a given graph G can be embedded in a surface of a given genus K. The NP-complete SUBGRAPH ISOMORPHISM problem ([GT48] in [G&J]) asks whether the vertices of a given graph H can be mapped one-to-one into the vertices of a second graph G, in such a way that if vertices u and v are adjacent in H, then their images are adjacent in G. An incomplete list of other NP-complete embedding problems from [G&J] would include CLIQUE [GT19], HAMILTONIAN CIRCUIT [GT37], BANDWIDTH [GT40], OPTIMAL LINEAR ARRANGEMENT [GT42], GRAPH HOMOMORPHISM [GT52], ISOMORPHIC SPANNING TREE [ND8], and EDGE EMBEDDING ON A GRID [ND47].

In this column I present a number of complexity results that have been proved about embedding problems since the appearance of [G&J]. I shall begin by surveying some new developments concerning problems from our original list, and conclude with a series of nine *new* NP-hard problems, and an ''Open Problem of the Month.''

The update on old problems will be sandwiched between some bad news and some good news about HAMILTONIAN CIRCUIT [GT37]. In the last column [Dec. 1981], I mentioned the new polynomial time algorithms for finding maximum weight embeddings of cliques, that were guaranteed of success so long as the host graph was perfect [8]. The HAMILTONIAN CIRCUIT problem is not so lucky. A *grid graph* is a very restricted form of perfect graph. For vertices it has a set of points in the plane with integer coordinates, and its edge set consists precisely of those pairs of vertices that are *neighbors*, i.e., that agree in one component and disagree by exactly 1 in the other. Itai *et al.* [12] have recently shown that the HAMILTONIAN CIRCUIT problem remains NP-complete even when restricted to grid graphs. An analogous result holds for HAMILTONIAN PATH [GT39].

On the positive side, polynomial time algorithms have been found for important special cases of the BANDWIDTH [GT40] and MINIMUM CUT LINEAR ARRANGEMENT [GT44] problems. Recall that both these problems concern embedding functions $f: V \rightarrow Z$. The former problem looks for embeddings f that satisfy

$$\max \{ | f(u) - f(v) | : \{u,v\} \in E \} \leq K;$$

the latter for embeddings that satisfy

$$\max \{ | cut(j) | : j \in Z \} \leq K,$$

where $cut(j)$ is the set of all edges $\{u,v\}$ such that $f(u) \leq j < f(v)$. J. B. Saxe [23] has shown that BANDWIDTH can be solved in polynomial time for any fixed K using dynamic programming, while Gurari and Sudborough [10] have shown the same for MINIMUM CUT LINEAR ARRANGEMENT (and improved upon Saxe's algorithm for BANDWIDTH [9]).

In addition, Monien and Sudborough [16,17] and Chung *et al.* [2] have discovered that a number of other NP-complete problems become solvable in polynomial time if restricted to graphs with bandwidth bounded by K, for any fixed K. Examples of such problems are VERTEX COVER [GT1], GRAPH 3-COLORABILITY [GT4], and, to return to our starting point, HAMILTONIAN CIRCUIT. If an embedding function f is provided along with the graph, these problems can be solved in polynomial time for bandwidths as large as $\log_2 |V|$. References [2,16,17] also extend the notion of ''bandwidth'' to objects other than graphs, and study the relation between bandwidth restrictions and classes of problems defined by simultaneous time and space bounds.

Let us now turn to this column's nine new problems. Each problem will be presented in the format used by [G&J], with related results appearing in an associated *Comment* section. Each problem is NP-hard, and may be assumed to be NP-complete unless a statement to the contrary is found in the associated *Comment*.

[1] GRAPH ENCODABILITY

INSTANCE: Directed graphs $G = (V, A)$ and $H = (V', A')$.

QUESTION: Is there an encoding of G in H, i.e., a one-to-one function $f\colon V \to V'$ such that, for each arc $(u,v) \in A$, there exists a directed path from $f(u)$ to $f(v)$ in H.

Reference. Rosenberg [20]. Transformation from 3SAT.

Comment. Note that this problem differs from the directed SUBGRAPH HOMEOMORPHISM problem [Dec. 1981] in that the paths corresponding to edges need not be vertex-disjoint. It also differs in that it is solvable in polynomial time for any fixed G, and the undirected version is solvable in polynomial time even if G is not fixed (both observations left as exercises to the reader). A less straightforward observation, due to Rosenberg *et al.* [21], is that the directed version is also solvable in polynomial time if G is a directed path. NP-completeness for *un*directed graphs can be obtained if one considers the weighted version of the problem, where each edge of H has an associated weight, and the cost of an encoding is taken to be the sum (alternatively, the maximum) of the costs of the paths in H corresponding to edges of G. NP-completeness holds even if all weights are equal. For details of these results, as well as even more general versions of the problem see [20].

The above problem arises when one wishes to simulate one data structure with another. In the weighted versions one is trying to minimize the expected (or maximum) time for simulating the use of a pointer. One drawback to this simulation scheme is the large amount of storage space that would be required to store the induced mapping from arcs of G to paths of H. The next problem deals with a plan for designing encodings that can be represented more concisely.

[2] UNIFORM GRAPH ENCODABILITY

INSTANCE: Directed graphs $G = (V, A)$ and $H = (V', A')$, sets L, L' of *labels* with $|L|$ = maximum out-degree of G and $|L'|$ = maximum out-degree of H, *labeling functions* $h\colon A \to L$ and $h'\colon A' \to L'$ such that no two arcs with the same initial vertex get the same label.

QUESTION: Is there a *uniform encoding* of G in H, i.e., a pair of one-to-one maps $f\colon V \to V'$ and $e\colon A \to \{\text{paths in } H\}$ such that (i) for all $a = (u,v)$ in A, $e(a)$ is a path from $f(u)$ to $f(v)$ and (ii) if a_1 and a_2 are arcs in A with the same label, then the sequence of labels on the path $e(a_1)$ is the same as the sequence on the path $e(a_2)$?

Reference. Rosenberg *et al.* [21]. Transformation from FINITE STATE AU-

TOMATA INTERSECTION [AL6].

Comment. PSPACE-complete, even if H has maximum out-degree 3 and G is a directed path. If the images of arcs are required to be *simple* paths (no repeated vertices), then the problem is merely NP-complete, in both the general case and the case where G is a directed path. If H is a connected ''path-regular'' graph (as defined in [21]), the problem can be solved in polynomial time given any fixed bound on the maximum out-degree in G [20,21].

The following variant on the GRAPH HOMOMORPHISM problem [GT52] claims its motivation from formal language theory. More specifically, it relates to the notion, due to Wood [31], of defining a family of related grammars in terms of homomorphisms to a ''master grammar.''

[3] SUBGRAPH HOMOMORPHISM

INSTANCE: Graphs $G = (V_1, E_1)$, $H = (V_2, E_2)$, with self-loops allowed but no multiple edges.

QUESTION: Is there a homomorphism from G to a subgraph of H, i.e., a function $f\colon V_1 \to V_2$ such that if $\{u,v\} \in E_1$ then $\{f(u), f(v)\} \in E_2$?

Reference. Maurer *et al.* [15]. Transformation from GRAPH 3-COLORABILITY.

Comment. This problem differs from GRAPH HOMOMORPHISM [GT52] mainly in that here the image of G need not be isomorphic to H but instead can be a proper subgraph (if H is loop-free the problems are otherwise equivalent). As with SUBGRAPH HOMEOMORPHISM [Dec. 1981], we may ask about the complexity of the problem when the graph H is fixed. If H is an odd cycle or a complete graph on 3 or more vertices, the derived problem is NP-complete. If H is bipartite or contains a self-loop, the derived problem is solvable in polynomial time. It is a believable conjecture that these are the *only* fixed H for which the problem is solvable in polynomial time [15]. The corresponding problem for directed graphs does not appear to yield such a straightforward classification scheme. If H contains a self-loop or is a path, cycle, or transitive tournament, the problem is in P. However, if H is an odd cycle with one additional arc (a back-arc between two adjacent vertices in the cycle) the problem becomes NP-complete. All directed graphs H with 3 or fewer vertices have been classified on a case-by-case basis, but no general pattern has emerged [15].

The next problem provides us with a transition from questions of embedding graphs in other graphs to questions of a more geometric nature, since it can be viewed in either guise.

[4] EMBEDDING DIMENSION

INSTANCE: Graph $G = (V, E)$, positive integer K.

QUESTION: Does G have embedding dimension K or less, i.e., is there a one-to-one function $f: V \rightarrow \{0,1,2\}^K$ such that, for all $u,v \in V$, $\{u,v\} \in E$ if and only if $f(u)$ and $f(v)$ differ by at most 1 in any component?

Reference. Dewdney [3]. Transformation from VERTEX COVER.

Comment. Solvable in polynomial time for any fixed K, since there are less than $|V|^{3^K}$ possible embeddings to consider. The graphical interpretation views $\{0,1,2\}^K$ as the vertex set of a graph, with an edge between each pair of vertices if their maximum component-wise difference is 1, and asks if G is an induced subgraph. The geometric version views $\{0,1,2\}^K$ as a metric space under the L_∞ metric, and asks if there is an isometric embedding of G into this space, when G is viewed as a metric space under the metric d, defined as follows: $d(u,v) = 0$ if $u = v$, $d(u,v) = 1$ if $\{u,v\}$ is an edge, and $d(u,v) = 2$ otherwise. The embedding dimension of a graph with n vertices must be n or less, and the dimensions of a number of special graphs have been determined [3]. Also see [3] for a discussion of possible extensions of the idea of embedding dimension to other metrics and to base spaces other than $\{0,1,2\}$. (Note that the similarity of this problem to that of NESETRIL-RÖDL DIMENSION [GT62] is only superficial.)

The next problem considers embeddings into Euclidean spaces, and makes the job of embedding somewhat easier by not prespecifying the distance between a pair of vertices unless they are *adjacent* in G.

[5] WEIGHTED GRAPH EMBEDDABILITY

INSTANCE: Graph $G = (V, E)$, *weight* $w(u,v) \in Z^+$ for each edge $\{u,v\} \in E$, and a positive integer *dimension* K.

QUESTION: Is there a function $f: V \rightarrow R^K$ (not necessarily one-to-one), such that for all $\{u,v\} \in E$, the Euclidean distance between f(u) and f(v) equals the weight $w(u,v)$?

Reference. Saxe [24,25]. Transformation from 3SAT.

Comment. NP-hard for any fixed $K \geq 1$, even if all edge weights are equal [24,25] (a less restricted version was independently shown NP-hard by Y. Yemeni in [33]). The problem is only known to be in NP for $K = 1$. Remains NP-hard for any fixed K even if we are given one embedding and asked whether a second (non-isomorphic) embedding exists. If G is a complete graph, the problem is solvable in polynomial time via an algoithm presented in [24] and attributed to M. I. Shamos. This algorithm works even if the weights are arbitrary positive real numbers, so long as your computer can perform arithmetic operations on reals, including comparisons and extractions of square roots, with each such operation taking constant time. (If you have such a computer, please let me know.) Other solvable special cases are considered in [33], which concentrates on ''rigid'' embeddings.

An interesting variant on the above problem is obtained if one requires f to be one-to-one (all vertices map to distinct locations). For $K = 1$ and all weights equal, the problem now becomes trivial (only paths will do). For arbitrary weights the problem is open for all $K \geq 1$. If one in addition requires that each $f(v)$ be a point with integer coordinates, one can use the HAMILTONIAN PATH FOR GRID GRAPHS problem (as mentioned above) to derive NP-completeness results for $K = 3$ and equal weight, and for $K = 2$ and weights 1 and 2 [13]. The $K = 1$, arbitrary weight case remains open in this variant also.

Let us now turn to embeddings in which the distances between points are *not* prespecified, and the main constraint is that the images of edges are not allowed to intersect. Under these restrictions, questions of ''dimension'' are considerably simplified: only a one-vertex graph can be embedded in R^0, only a path in R^1, only planar graphs in R^2 (by definition), and all graphs can be embedded in R^3. The interesting questions thus revolve around the *quality* of the embedding. Each of the following three problems considers this from a different point of view.

[6] MINIMUM AREA EMBEDDING OF PLANAR GRAPHS

INSTANCE: Planar graph $G = (V, E)$ with vertex degrees bounded by 4, positive integer K.

QUESTION: Is there a *grid embedding* of G which can be bounded by a rectangle of area K or less? A grid embedding is a function f that sends vertices of G to distinct grid points (points with integer coordinates), and edges to non-intersecting paths made up of grid segments (line segments between pairs of grid points that are adjacent either horizontally or vertically). Note that the vertex degrees must be bounded by 4 if G is to be embedded in the grid.

Reference. Dolev and Trickey [4]. Transformation from 3-PARTITION.

Comment. NP-complete even if G is a forest. Open for trees, and, as far as I know, for connected graphs. If a planar representation is given along with G, and the grid embedding is required to preserve the regions of this representation, then the problem *is* NP-complete for connected graphs, as proved by J. A. Storer [26,27].

The motivation for minimizing area comes from applications to VLSI (Very Large Scale Integration) circuit design. Given that wire widths are fixed, the less area a circuit takes up, the easier it is to produce reliably. A number of approximation algorithms for the area minimization problem have been proposed. Valiant [29] gives an algorithm that can embed any tree in $O(|V| + |E|)$ area, and Dolev and Trickey extend this result to outerplanar graphs and graphs that are ‘‘nearly’’ outerplanar. Note that there are planar graphs for which quadratic area is required [29] and if all leaves must be on the boundary of the enclosing rectangle, there are trees that require $|V| \log |V|$ area [1]. Approximation algorithms (but as yet no NP-completeness results) have also been obtained for the variant of this problem (again motivated by VLSI) where edges *are* allowed to cross (but only at right angles and two at a time) [14,29]. Note that in this variant the restriction that G be planar can be dropped, but not the bound on vertex degree.

A second motivation for attempting to minimize area in grid embeddings is that this corresponds (roughly) to minimizing total edge length, an age-old criterion for layout ‘‘optimality’’. The problem of minimizing total edge length itself has been considered by two researchers. Storer [26,27] shows that it is NP-hard to minimize total edge length in region-preserving grid embeddings (and also considers the problem of minimizing the total number of bends occurring in the paths representing edges in such embeddings). If regions need not be preserved, Woods [32] shows that it is NP-hard to minimize total edge length in embeddings where, although all vertices and ‘‘bends’’ must occur at grid points, the individual segments making up an edge need not be strictly horizontal or vertical. I have not yet seen a proof for the case of standard grid embeddings when regions need not be preserved. The question of minimizing *maximum* edge length in a grid embedding has also been studied [18], but here, too, complexity results are absent. The next problem concerns itself with *weighted* sums of edge lengths when the restriction to grid embeddings is traded in for a different one.

[7] WEIGHTED TREE LAYOUT WITH FIXED LEAVES

INSTANCE: Tree $T = (V, E)$, weight $w(e) \in Z^+$ for each edge e, *location* $f(v) \in Z^2$ for each leaf vertex in V, positive integer K.

QUESTION: Is there an embedding of T in R^2 that (i) sends each leaf v to its location f(v) but can send non-leaf vertices anywhere, (ii) sends each edge e to a curve $c(e)$ connecting its endpoints, with no two such curves intersecting except at a common endpoint, and (iii) satisfies $\sum_{e \in E} w(e) \cdot \text{length}[c(e)] \le K$, where curve length is computed according to the Euclidean metric?

Reference. Fischer and Paterson [5]. Transformation from PLANAR 3SAT.

Comment. NP-hard even if all weights are equal. Not known to be in NP. Variant in which curves representing edges *are* allowed to cross (and hence may be assumed to have length equal to the distance between their endpoints) can be solved in polynomial time if the L_1 (rectilinear) distance metric is used, even with arbitrary weights [5].

The motivation for the above problem is, once again, VLSI. The motivations for the next problem are mainly aesthetic.

[8] EUMORPHOUS TREE LAYOUT

INSTANCE: Rooted binary tree $T = (V, E)$, positive integer K.

QUESTION: Is there an assignment of the vertices of T to distinct grid points so that all can be enclosed within a rectangle of width K or less, and the following properties are satisfied?

(1) All vertices with level i in T have i as their y-coordinate, and must be at least two units apart,

(2) A rightchild is to the right of its parent and a leftchild is to the left of its parent, with the parent centered on both if it has both,

(3) If the edges of T are drawn as straight lines, no two cross each other, and

(4) If T_1 and T_2 are isomorphic subtrees of T, then their images in the embedding are identical (up to translation).

Reference. Supowit and Reingold [28]. Transformation from 3SAT.

Comment. The problem of producing a layout whose width is guaranteed to be no worse than 25/24 times optimal is also NP-hard. However, if the restriction that x-coordinates be integers is dropped (all other constraints being retained), the problem can be solved by linear programming and hence in polynomial time. Heuristics for the integral case (with and without condition (iv)) are described in [19,30] and analyzed in [19,28]. *Eumorphous* is from the ancient Greek and means ‘‘well-shaped’’.

Our final problem concerns one measure of how ‘‘near-planar’’ a given nonplanar graph is. It was mentioned as open in [G&J] and would have been my ‘‘Open Problem of the Month,’’ if only I had managed to write this column a bit sooner.

[9] CROSSING NUMBER

INSTANCE: Graph $G = (V, E)$, positive integer K.

QUESTION: Is there an embedding of G in the plane such that no more than two edges intersect at any one point (other than a common endpoint), and there are at most K of these crossings?

Reference. Garey and Johnson [6]. Transformation from OPTIMAL LINEAR ARRANGEMENT.

Comment. Solvable in polynomial time for any fixed K. Also NP-complete is the following variant, which was used as an intermediate step in the proof of the main result: Given a bipartite graph $G = (V_1, V_2, E)$ and K, can G be embedded in a unit square so that all vertices from V_1 are on the North boundary, all vertices from V_2 are on the South boundary, and the number of crossings is K or less?

As hinted above, it is my intention to feature at least one open problem in each column. Readers should be forewarned that there is no guarantee that a solution to such a problem will yield an automatic addition to their publication lists. I am, of course, eager to report the answers to questions raised in this column, but there is no guarantee that a journal will consider such results publishable by themselves. Thus readers may have to settle for having their results publicized rather than published. I hope this will be viewed as reward enough.

Although CROSSING NUMBER has been preempted, there remains another, equally interesting embedding problem that was mentioned in passing in [G&J] and remains open today, and this will be our highlighted problem for the current issue:

[OPEN] GRAPH THICKNESS

INSTANCE: Graph $G = (V, E)$, positive integer K.

QUESTION: Does G have *thickness* K or less, i.e., is there a partition of E into sets $E_1, ..., E_K$ such that each of the graphs $G_K = (V, E_K)$ is planar?

Comment. This problem is of mathematical interest because, like the open GRAPH GENUS problem and the NP-complete CROSSING NUMBER problem, it concerns a measure of ''nearness'' to planarity that has been well-studied by graph theorists. Results on the thickness of various classes of graphs are summarized in [11, pp. 120–121]. Practical applications are a bit more tenuous, although a case might be made for relevance to VLSI layout, and graph thickness does enter into a scheme for testing printed circuits, as described in [7].

In the next issue, I hope to continue in a VLSI-related vein by discussing a number of problems concerned with routing and the covering, packing, and partitioning of geometric objects.

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