Published in J. ALGORITHMS 3, 381-395 (1982)

The NP-Completeness Column: An Ongoing Guide

DAVID S. JOHNSON

Bell Laboratories, Murray Hill, New Jersey 07974

This is the fifth edition of a quarterly column the purpose of which is to provide continuing coverage of new developments in the theory of NP-completeness. The presentation is modeled on that used by M. R. Garey and myself in our book ''Computers and Intractability: A Guide to the Theory of NP-Completeness,'' W. H. Freeman & Co., San Francisco, 1979 (hereinafter referred to as ''[G&J]''; previous columns will be referred to by their dates). A background equivalent to that provided by [G&J] is assumed, and, when appropriate, cross-references will be given to that book and the list of problems (NP-complete and harder) presented there. Readers who have results they would like mentioned (NP-hardness, PSPACE-hardness, polynomial-time-solvability, etc.), or open problems they would like publicized, should send them to David S. Johnson, Room 2C-355, Bell Laboratories, Murray Hill, NJ 07974, including details, or at least sketches, of any new proofs (full papers are preferred). In the case of unpublished results, please state explicitly that you would like the results to be mentioned in the column. Comments and corrections are also welcome. For more details on the nature of the column and the form of desired submissions, see the December 1981 issue of this Journal.

1. INTRODUCTION

This ''ongoing guide'' celebrates its first anniversary with the current column, and so it would seem appropriate for me to provide a status report. Responses to my calls for new results related to NP-completeness continue to arrive at a rate of about 5 a month, with perhaps an equal number of results showing up unheralded in the literature. Among these there is a certain amount of duplication, along with a certain number of results that are either too trivial, too obscure, or too incorrect for mention in the column. The net result, however, is a continuing growth in the size of the backlog (currently there are about 100 items in my ''to be covered'' file).

Part of the blame for this growth falls on my shoulders. In hopes of making the column interesting to read, I have been including a lot more in the way of continuity and motivation than was possible in the condensed style of [G&J], and consequently have been able to cover fewer results in a given number of pages. I welcome comments and suggestions on this or any other matter of content or style. To those (too numerous to mention) whose advice and assistance have helped get me through this first year, my grateful thanks.

The subject of this month's column is *routing problems*, where I take the broad view that a ''routing problem'' is any problem that has to do with the interconnection of points. Thus we could be concerned with anything from connecting logical gates in an integrated circuit design to finding Hamiltonian paths in graphs, and I can survey a wide range of results, with the usual miscellany of applications (both real and imaginary). The results to be covered do, however, break nicely into two groups: Section 2 is devoted to problems of interconnecting vertices in graphs, and Section 3 is devoted to problems of interconnecting points in the plane. The latter section includes the resolution of one of our past ''Open Problems of the Month,'' plus a pseudo-resolution of one of the remaining open problems from [G&J].

2. ROUTING IN GRAPHS

One of the most famous vertex-interconnection problems covered in [G&J] is STEINER TREE IN GRAPHS [ND12]: Given a graph $G = (V,E)$, a subset $R \subseteq V$ of *required* vertices, a non-negative integer weight $w(e)$ for each edge $e \in E$, and a bound $B > 0$, is there a subtree of G that includes all vertices of R and whose total edge weight is B or less? New results relating to this problem come from two sources. First, we have some new polynomial time solvable subcases. STEINER TREE was known to be NP-complete even if all weights are equal and G is planar. However, if we restrict ourselves to *outerplanar* graphs (planar graphs that can be embedded in the plane so that the boundary of the external region is a Hamiltonian circuit of the graph), Wald and Colbourn [45] have shown that minimum Steiner trees can be found in linear time, even if arbitrary weights are allowed. In [46], they generalize this to the case of *partial 2-trees*. (A *2-tree* is obtained by starting with a triangle and repeatedly choosing an edge and adding a new vertex adjacent to both its endpoints. A *partial* 2-tree is obtained by deleting edges from a 2-tree.)

The second STEINER TREE development concerns a newly discovered NP-complete subcase for the equal-weight problem: the set of *cubical* graphs [9]. A graph is cubical if it is an N-cube for some $N > 0$, where in an N-cube the vertices are all the binary N-tuples and two vertices are adjacent if and only if they differ in exactly one component. Given the regularity of the structure of these graphs, it is surprising that a problem could be NP-complete when restricted to them. However, the fact that the set R of required vertices may be chosen arbitrarily turns out to provide the degrees of freedom necessary for an NP-completeness proof. Creationists in the audience will no doubt applaud this result, as the STEINER TREE problem for cubical graphs must be solved if one is to construct an evolutionary tree of ‘‘maximum parsimony’’ for a set of species (even if one is concerned with ‘‘just two amino acid triples’’) [9].

Let us now turn from updating entries in [G&J] to some new problems involving vertex interconnection. We begin with three variants on the standard problem of finding a path connecting a vertex s to a vertex t, all of which can be related to problems of mass transit.

[1] BUS ROUTING

INSTANCE: Acyclic directed graph $G = (V,A)$, two specified vertices $s,t \in V$, a non-negative integer weight $w(a)$ for each arc a in the transitive closure A^* of A, and a bound $B > 0$.

QUESTION: Is there a path P in G from s to t such that, if P^* is the transitive closure of the arcs in P, $\sum_{a \in P^*} w(a) \geq B$?

Reference. Röck [33]. Transformation from HAMILTONIAN PATH.

Comment. Remains NP-complete even if G consists of a rectangular grid, with s the northwest corner, t the southeast corner, and all arcs directed either south or east. The application to bus routing occurs when the weight of an arc (u,v) corresponds to the number of potential passengers who want to go from vertex u to vertex v. If (u,v) is in the transitive closure of the path taken by the bus, these passengers can be accommodated. If we want to maximize the number of passengers accommodated (and are willing to assume an infinite-capacity bus) we get the above problem. If we want to *minimize* the number of passengers accommodated we get a quite different problem, but it is still NP-complete for the grid [33]. Both variants are solvable in polynomial time for the grid if the weighting function decomposes into a sum of two terms, each depending only on one of the endpoints of the arc and the rectilinear distance to the other [32]. Note that if we replace $\sum_{a \in P^*} w(a)$ by $\sum_{a \in P} w(a)$, we can use dynamic programming to solve both variants in polynomial time, even for arbitrary weights and arbitrary acyclic directed graphs (the graphs need not even be acyclic in the minimization variant).

The next problem can also be viewed as a bus routing problem, one in which the passengers are particularly demanding.

[2] ROUTING WITH ‘‘MUST PAIRS’’

INSTANCE: Acyclic directed graph $G = (V,A)$, two specified vertices $s,t \in V$, and a subset M of the transitive closure A^* of A, with the elements of M called *must-pairs*.

QUESTION: Is there a path P in G from s to t such that for each must-pair $(u,v) \in M$, if P passes through u then it also passes through v?

Reference. Ntafos and Hakimi [23]. Transformation from 3SAT.

Comment. This problem was actually motivated by software testing, in which case G is the flowgraph of a program. As such it is related to the problem PATH WITH FORBIDDEN PAIRS [GT54] studied in [10]. It remains NP-complete even if G arises from a ''structured'' program [24]. For related results concerning the covering of graphs by paths in the presence of ''required pairs,'' ''impossible paths,'' etc., see [23,24].

Relating our last $s-t$ routing problem to busing will take real ingenuity. Perhaps it arises in the case of city-owned bus companies, where, in contract disputes, it is quite common for bus drivers to demand *parity*.

[3] EVEN LENGTH PATH

INSTANCE: Directed graph $G = (V,A)$, specified vertices $s,t \in V$.

QUESTION: Is there a simple path P from s to t in G which contains an *even* number of arcs?

Reference. LaPaugh and Papadimitriou [17]. Transformation from DIRECTED SUBGRAPH HOMEOMORPHISM [Dec. 1981].

Comment. The problem is solvable in linear time for undirected graphs [17]. In fact, for edge-weighted undirected graphs, a *minimum length* even path can be found in polynomial time by a clever application of matching algorithms [7]. The same results apply if ''even'' is replaced by ''odd.'' Back in the directed case, suppose the arcs of G are 2-colored. If we ask that the path not only contain an even number of arcs, but also that the arcs alternate in color along the path, we again have an NP-complete problem [44], although this variant too is solvable in polynomial time if G is an undirected graph [44]. It is also solvable in polynomial time if either of the two color classes constitutes a matching.

I conclude this section with a problem that, while it does not involve interconnection of vertices *per se*, does concern an old stand-by of graph routers: the Euler tour. In this problem, we trade in the infinite-capacity bus of problem [1] for a fixed-capacity delivery van.

[4] DELIVERY VAN ROUTING

INSTANCE: Directed graph $G = (V,A)$, each arc a of which has an associated (possibly negative) integer weight $w(a)$, and a vertex $s \in V$.

QUESTION: Is there a permutation $P = (a_1, \ldots, a_m)$ of the arcs in A such that (i) for all i, $1 \le i < m$, the head of a_i coincides with the tail of a_{i+1} (i.e, P is an Euler tour of G), (ii) s is the tail of a_1 and the head of a_m, and (iii) for all j, $1 \le j \le m$, $\sum_{i=1}^{j} w(a_i) \le 0$?

Reference. Boers [4]. Transformation from 3SAT.

Comment. Remains NP-complete even if each vertex of G has an in-degree equal to its out-degree (and hence G has at least one Euler tour), and even if, in addition, all weights are from the set {-1,1}. Easily solvable for arbitrary graphs if all weights are 0 or less, for in this case all we are asking is whether an Euler tour exists. The interpretation in terms of delivery vans is as follows: A fully-loaded van is to start at point s and traverse all the streets of the city (arcs of the graph) making pick-ups and deliveries on each block. Since it starts out completely full, its route must be designed so that at no point along the way has it ever picked up more than it has delivered. The restriction to Euler tours corresponds to the admirable desire to minimize gasoline usage.

3. Interconnection of Points in the Plane

Although most of the problems in this section are motivated by applications to circuit design (both VLSI chips and printed circuit boards), we shall begin with a less ‘‘technology-driven’’ result. The following is an ancient geometric puzzle whose only significant application is to the traditional problem of conserving ammunition during target practice.

[5] EUCLIDEAN POINT COVER

INSTANCE: A set $P = \{ p_1, p_2, \ldots, p_n \}$ of integer-coordinate points in the plane and a positive integer K.

QUESTION: Is there a collection of K or fewer straight lines such that every point in P is contained in at least one of the lines?

Reference. Proved independently by Megiddo and Tamir [22] and van Emde Boas [43]. Transformation from 3SAT.

Comment. NP-complete in the strong sense. Solvable in polynomial time by bipartite matching techniques if each line must be either horizontal or vertical. Solvable trivially if no three points are collinear. The dual EUCLIDEAN LINE

COVER problem, in which we are given a set of (infinite) straight lines and are asked if there exist K points such that each line contains one of the points, is also NP-complete, but solvable in polynomial time if there are no intersections involving three or more lines [22].

We now turn to problems motivated by VLSI design and circuit wiring. In the literature on these problems one is faced with a bewildering array of routing models; in this column we consider only those for which complexity results have been proven (and which may only approximate ''real world'' routing situations). I have organized the discussion around four basic problems, each of which will be introduced in terms of some variant of the ''rectilinear'' routing model, with results for other models summarized in the Comments sections.

For the purposes of this column, the rectilinear model assumes that routing takes place in a standard *rectilinear grid*, where each integer-coordinate point is a *grid point*, connected by length-1 *grid segments* to its four immediate neighbors. The grid is a simple way of enforcing the requirement that wires and components be separated by certain minimum distances (this requirement is imposed by fabrication constraints and the electrical properties of the circuits). The objects (*terminals*) to be interconnected will normally be grid points, with a *net* being a set of terminals that are all to be interconnected with each other.

The interconnection will normally be done by collections of grid segments. There are three basic ways in which two such rectilinear ''interconnection patterns'' can violate the separation requirement. They can have an *overlap* (a common grid segment), an *intersection* (a common point), or an *orthogonal intersection* (a common point that joins two vertical grid segments in one pattern and two horizontal segments in the other). The extent to which these possibilities are forbidden depends upon the design methodology imposed.

For now, I shall mention three alternatives, each differing from the others in the number of distinct ''layers'' of wiring allowed. If *no* intersections or overlaps are allowed, we have the problem of ''planar'' routing, which corresponds to routing in a single layer. If only orthogonal intersections are allowed, we have a two-layer problem known as ''reserved layer'' routing, where one layer is reserved for horizontal grid segments and the other for vertical ones, with direction (and level) changes occurring only at grid points (the grid points at which such changes occur are called *vias*). Finally, if each interconnection pattern must be assigned one of K layers, cannot intersect at all with a pattern on its own layer, but can overlap and intersect arbitrarily with patterns on other layers, we have what is called ''K-layer via-free'' routing. We begin with a problem that is stated in terms of the first methodology, but for which there are results relevant to all three.

[6] PLANAR INTERCONNECTION ON A GRID

INSTANCE: A set $P = \{(p_1,q_1),...,(p_n,q_n)\}$ of disjoint *pairs* of integer-coordinate grid points.

QUESTION: Is there a way of connecting each pair of grid points in P by a path made up of grid segments, so that no two of the paths intersect?

Reference. Proved independently by Richards [31] and Kramer and van Leeuwen [13]. Transformation from PLANAR 3SAT.

Comment. NP-complete in the strong sense. Remains so even if orthogonal intersections are allowed [13]. It is solvable in polynomial time if no path is allowed to have more than one bend [28]. However, if no routing is possible in this case, then the problem of finding a maximum number of pairs that can be simultaneously routed is NP-hard, as is the problem of determining whether there is a 2-layer via-free routing [28]. Note that the original grid problem can be viewed as the graph problem DISJOINT CONNECTING PATHS [ND40], restricted to a very special class of planar, maximum degree 4 graphs. In [31] it is shown that this graph problem remains NP-complete even for planar, maximum degree 3 graphs.

[7] TWO-LAYER MULTIPLE MODULE ROUTING

INSTANCE: Collection $\{M_1,...,M_n\}$ of non-intersecting *modules* (rectangles specified by their four integer-coordinate corners), set P of *terminals* (grid points on the perimeters of the modules), partition of P into *nets* $N_1,...,N_m$, and an integer bound B.

QUESTION: Is there a (reserved layer) routing for the nets that can be enclosed within a rectangle of area B, i.e., is there a set of grid segments $S = T_1 \cup \cdots \cup T_m$ and a rectangle R such that (1) all the modules and all the grid segments in S are included in rectangle R, and none of the grid segments is contained in any of the modules, (2) for each i, $1 \le i \le m$, the segments in T_i make up a tree whose leaves are the points in net N_i (T_i is an *interconnection pattern* for N_i), and (3) no two trees T_i and T_j have any *non-orthogonal* intersections? See Fig. 1, in which each terminal is represented by a ‘‘•’’, as are the junction points within individual interconnection patterns.

Reference. Szymanski [36]. Transformation from 3SAT.

Comment. Remains NP-complete even if there are only two modules and each net contains only two terminals [37]. Approximation algorithms with provable performance guarantees have been devised for this restricted version in the case where the two modules have their sides aligned vertically [3]. If there is only *one* module and only 2-terminal nets are allowed, the problem can be solved in polynomial time [15,16].

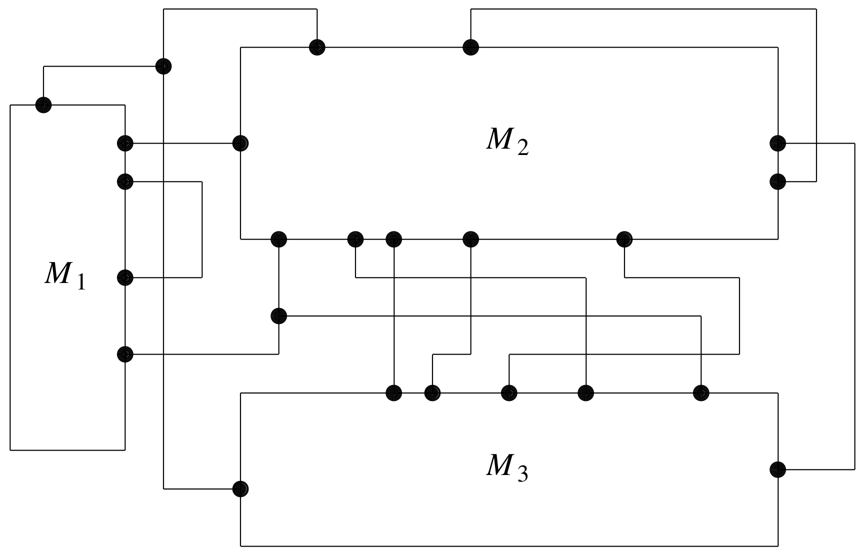

FIG 1. TWO-LAYER MULTIPLE MODULE ROUTING.

[8] TWO-LAYER CHANNEL ROUTING

INSTANCE: Positive integer W, set of grid points $P = \{(a_i, W) : 1 \le i \le p\} \cup \{(b_j, 0) : 1 \le i \le q\}$, and a partition of P into *nets* $N_1, \ldots, N_m$.

QUESTION: Is there a (reserved layer) two-layer channel routing for the nets, i.e., a set of grid segments $S = T_1 \cup \cdots \cup T_m$ where all the grid segments in S are included in the region between the lines $y = 0$ and $y = W$ (with none *from* those two lines), and properties (2) and (3) of the previous problem definition hold?

Reference. Szymanski [36]. Transformation from 3SAT.

Comment. This is essentially a special case of the previous problem, with the (a_i, W)'s viewed as terminals on one side of a component, the $(b_j, 0)$'s as terminals on the facing side of an adjacent component, and the interconnecting wires all constrained to lie in the region between the two components. It arises in a common approach to breaking up the general problem into more ''manageable'' subproblems [12]. I use ''W'' as an abbreviation for the ''height'' of the channel, since circuit designers normally call this the ''channel width'' (this is only half as bad an abuse of rotation as that used by computer scientists for describing ''trees''). Minimizing W corresponds to minimizing the ''separation'' required between the two components.

The problem remains NP-complete even if all nets consist of just two points, one of the form (a_i, W) and one of the form $(b_j, 0)$ [37]. It is also NP-complete for nets of this form if each T_i is only allowed to contain one horizontal line segment (sequence of horizontal grid segments) [15,16]. If, however, no two terminals share a common x-component, then the problem reduces to that of coloring an interval graph, and can be solved in polynomial time [12]. Even without this constraint, the problem is solvable in polynomial time (by dynamic programming) for any fixed value of W, so long as the maximum number of terminals per net is also bounded. Approximation algorithms for variants of the problem are presented in [32]. Polynomial time algorithms are presented in [25] for various restricted versions of routing in a ''T-shaped'' channel.

When one turns to the special case of channel routing known as *river routing*, one finds polynomial time solvable problems in profusion. In this special case, the nets are of the form $\{(a_i, W), (b_i, -W)\}$, $1 \le i \le m$ and both $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \cdots < b_m$. The river-routing problem is solvable in polynomial time, not only in the case of the reserved layer model [6], but also in the single-layer case (*no* intersections are allowed between trees T_i) [6], even when the restriction to rectilinear wires is weakened in various ways (while maintaining the constraint that two wires must be at least distance 1 apart) [6,18,34]. Also solvable in polynomial time for these and other models of river routing are the related problems of finding an ''offset'' of the upper and lower components [6,34], or, more generally, a horizontal spreading-apart of their individual terminals [18], so as to minimize separation, or so as to minimize the size of the minimum rectangle that contains all the T_i. The single-layer river-routing problem of minimizing total wire length is solvable in polynomial time, even if arbitrary wires (separated by at least unit distance) are allowed [40].

For a discussion of channel routing, as seen from the ''real world'' point of view of the professional circuit designer, see [5].

[9] SINGLE ROW ROUTING

INSTANCE: Positive integers B and W, partition of the set of points $\{(iB, 0) : 0 \le i \le n\}$ into disjoint *nets* $N_1, \ldots, N_m$.

QUESTION: Is there a single row routing for the nets, i.e., a set of grid segments $S = T_1 \cup \cdots \cup T_m$ where all the grid segments in S are included in the rectangular region with corners at $(0, -W)$, $(0, W)$, $(nB, -W)$, and (nB, W), each T_i is an interconnection pattern for the corresponding net N_i, no two trees T_i and T_j intersect (even orthogonally), and for each i, no vertical line can be drawn that intersects two horizontal segments of T_i (except possibly at their endpoints)? See Fig. 2.

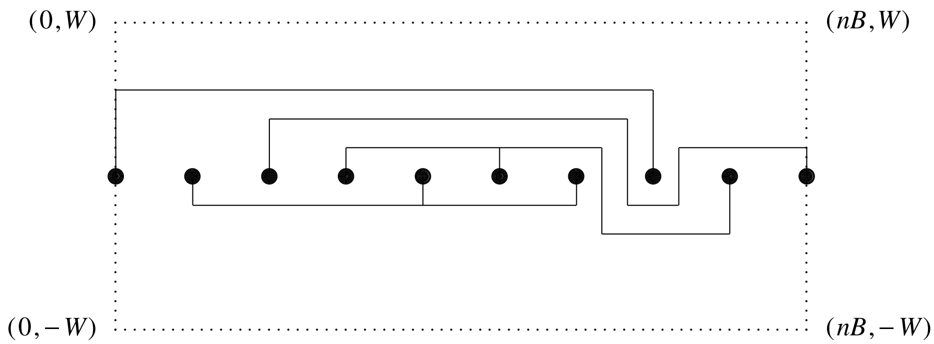

FIG 2. SINGLE ROW ROUTING.

Reference. Proved independently by Raghaven and Sahni [30] and Arnold [2]. Transformation from BANDWIDTH [GT40].

Comment. This problem arises in a approach introduced in [35] for the design of two-layer printed circuit boards (an older technology than VLSI), in which the ''layers'' correspond to the two sides of the board. The approach simplifies the general routing problem by placing all components in an array of rows and columns, in such a way that all connections are made either between elements in the same row or between elements in the same column, with one layer reserved for each type of connection. Minimizing W corresponds to minimizing the separation between rows; minimizing B corresponds to minimizing the separation between columns. The problem can be solved in polynomial time if B and W both exceed m, since in this case an interconnection pattern always exists (e.g., see [14,30,42]). If W exceeds m but B does not, then the problem is NP-complete, and remains so for any fixed value of $B \geq 2$, so long as single-terminal nets are allowed [2]. It is solvable in polynomial time for $B = 1$ [2]. If B exceeds m but W does not, then the problem remains NP-complete [2,30], but is solvable in polynomial time for any fixed W [2,29,30]. The problem of minimizing the number of times the nets cross the x-axis (the total number of ''inter-street crossings'') is NP-complete [42], as is the problem of minimizing the total number of ''bends'' in the interconnection patterns [30]. See [30,38,39] for complexity results relating to the problems of assigning components to rows and columns when the ''single row routing'' design approach is to be used.

Our next result can be viewed under the general rubric of K-layer via-free routing, but in it we abandon the restriction to rectilinear routes, and allow *arbitrary* non-intersecting curves to form the interconnection patterns in a given layer. Some ''easy'' initial observations can be made about this model [27]. Assuming that the interconnection patterns may lie anywhere in the plane, the problem is trivial, as the routing can always be done in just one layer (we could, for instance, draw a single curve for each net, connecting all its terminals and looping around any previously drawn curves). If, however, the terminals are all on the lines $y = W$ and $y = 0$ and the interconnections must all lie *between* these lines (as in channel routing), the problem includes GRAPH K-COLORABILITY for circle graphs [11] as a special case, and hence is NP-complete for arbitrary K. (If each net has two terminals, one on each line, the problem becomes equivalent to GRAPH K-COLORABILITY for permutation graphs, and hence is solvable in polynomial time [8]).

Suppose now that there is no constraint on where the interconnections may lie, but that nets are allowed to have terminals in common (up until now we have been assuming that the nets form a *partition* of the terminals). This last new wrinkle takes us fairly far away from VLSI design, but it does bring us close to a graph theoretic problem that was the ''Open Problem of the Month'' in the [Mar. 1982] column, and which has now been shown to be NP-complete:

[10] GRAPH THICKNESS

INSTANCE: Graph $G = (V,E)$, positive integer K.

QUESTION: Does G have *thickness* K or less, i.e., is there a partition of E into sets $E_1, ..., E_K$ such that each of the graphs $G_K = (V,E_K)$ is planar?

Reference. Mansfield [20,21]. Transformation from PLANAR 3SAT. The proof is a quite ingenious exercise in component design.

Comment. Corresponds to the above routing problem restricted to instances in which all nets contain exactly two terminals. Remains NP-complete even if K is fixed at 2. Solvable in polynomial time if $K = 1$, for then the problem is simply GRAPH PLANARITY.

In addition to GRAPH THICKNESS, another of our highlighted ''open problems'' has *almost* been solved. The problem is MINIMUM LENGTH TRIANGULATION [OPEN12], which, *apropos* of this section's topic, is yet another point-interconnection problem. In [19] an NP-completeness proof is presented for the following variant, whose ''variance'' from the original problem may, at first glance, be hard to detect.

[11] MINIMUM LENGTH PSEUDO-TRIANGULATION

INSTANCE: Collection $C = \{(a_i,b_i): 1 \le i \le n\}$ of integer-coordinate points in the plane, and a positive integer B.

QUESTION: Is there a collection of line segments that divides the convex hull of C into triangles, in such a way that each segment has its endpoints in C, each point in C is covered by one of the segments, and the total length of the line segments under the discrete-Euclidean metric is at most B? (Recall that under this metric, the distance between two points (a_i, b_i) and (a_j, b_j) is defined to be $\lceil ((a_i - a_j)^2 + (b_i - b_j)^2)^{1/2} \rceil$.)

Reference. Lingas [19]. Transformation from PLANAR 3SAT.

Comment. The term ‘‘pseudo-triangulation’’ is used because, in an ordinary triangulation, it is required that each point of C must be a *vertex* of some triangle, not just be covered by an *edge* of some triangle. With that added constraint the problem becomes the still-open MINIMUM LENGTH TRIANGULATION problem mentioned above. The proof of the current result relies crucially on the possibility that instances may contain large sets of collinear points, sets which pseudo-triangulations (unlike ordinary triangulations) can use to their advantage.

Having lost one (almost two) of our open problems, it is time to introduce this column’s Open Problem of the Month, and so replenish our supply. For this, we return to two-layer routing, in order to consider one more routing methodology.

[OPEN] ‘‘UNRESTRICTED’’ TWO-LAYER CHANNEL ROUTING

INSTANCE: As in TWO-LAYER CHANNEL ROUTING

QUESTION: Is there a set $S = T_1 \cup \cdots \cup T_m$ of pairs (s,k), where s is a grid segment and $k \in \{1,2\}$ is the layer assignment for that segment, such that (1) all the grid segments occurring as first components in S lie in the region between the lines $y = W$ and $y = 0$ (with none from these lines), (2) for each i, $1 \le i \le m$, the segments occurring as first components in T_i make up a tree whose leaves are the points in net N_i, (3) although segments occurring in distinct T_i’s may overlap, no grid segment can occur in different T_i’s with the same layer assignment, and (4) if we call a grid point g a *via* when it is the common endpoint of grid segments with different layer assignments in a given tree T_i, then no via is the endpoint of grid segments occurring in two distinct trees T_i and T_j.

Comment. The routing rules used here allow for arbitrary use of the two layers, with the exception that, when a wire changes layers, it totally ‘‘uses up’’ the grid point (via) where that change takes place. This corresponds directly to what is actually possible in current two-layer technologies, although, in practice, *long* sections of overlap may introduce undesirable ‘‘crosstalk’’ into the circuit. There is one positive result relevant to these routing rules, that follows as a special case from results in [26]. Suppose we are given a *planar* interconnection pattern T_i for each net N_i. Then we can determine in polynomial time whether there exists an assignment of grid segments to layers within each T_i, so as to obtain a simultaneous routing of all the nets as specified by the current rules. Moreover, if a layer assignment exists, we can in polynomial time find such an assignment that minimizes the total number of vias used.

There is little else in the way of good news for this model. The channel routing results of [36,37] extend to the restricted version of the model in which non-orthogonal intersections are allowed but overlaps are forbidden, and one's natural suspicion is that the general problem is also NP-complete. This suspicion is bolstered by a result of Arnold [1], which shows NP-completeness for the variant of this problem in which (a) the set of grid segments is augmented to include the diagonals of the grid squares (intersecting diagonals may not be used in the same layer), and (b) each point in an interconnection pattern lies on a path within that pattern that connects two terminals and never reverses its horizontal direction. Both of these conditions seem to be required for the proof in [1], as does the unrealistic assumption that nets be allowed to contain arbitrarily many terminals. However, some of my colleagues have been inspired by this result to think a little bit harder about the main problem, and the reader should be forewarned that this ''open problem'' may not survive the 3-month delay until publication.

REFERENCES

1. P. B. ARNOLD, ''Complexity results for channel routing,'' Report No. TR-21-82, Center for Research in Computing Technology, Harvard University, Cambridge, Mass., 1982.
2. P. B. ARNOLD, ''Complexity results for single-row routing,'' Report No. TR-22-82, Center for Research in Computing Technology, Harvard University, Cambridge, Mass., 1982.
3. B. S. BAKER, A provably good algorithm for the two module routing problem, manuscript (1982).
4. H. BOERS, The ''race graph'': an NP-complete problem, manuscript (1982).
5. D. N. DEUTSCH, A dogleg channel router, *in* ''Proc. 13th Design Automation Conf.,'' pp. 425-433, Institute of Electrical and Electronics Engineers, New York, 1976.
6. D. DOLEV, K. KARPLUS, A. SIEGEL, A. STRONG, AND J. D. ULLMAN, Optimal wiring between rectangles, *in* ''Proceedings 13th Ann. ACM Symp. on Theory of Computing,'' pp. 312-317, Association for Computing Machinery, New York, 1981.
7. J. EDMONDS, unpublished results (1977).
8. S. EVEN, A. PNUELI, AND A. LEMPEL, Permutation graphs and transitive graphs, *J. Assoc. Comput. Mach.* **19** (1972), 400-410.
9. L. R. FOULDS AND R. L. GRAHAM, The Steiner problem in phylogeny is NP-complete, *Advances Appl. Math.* **3** (1982), 43-49.
10. H. N. GABOW, S. N. MAHESHWARI, AND L. OSTERWEIL, On two problems in the generation of program test paths, *IEEE Trans. Software Engrg.* **SE-2** (1976), 227-231.
11. M. R. GAREY, D. S. JOHNSON, G. L. MILLER, AND C. H. PAPADIMITRIOU, The complexity of coloring circular arcs and chords, *SIAM J. Algebraic and Discrete Methods* **1** (1980), 216-227.
12. A. HASHIMOTO AND J. STEVENS, Wire routing by optimizing channel assignment within large apertures, *in* ''Proc. 8th Design Automation Workshop,'' pp. 155-169, Institute of Electrical and Electronics Engineers, New York, 1971.
13. M. R. KRAMER AND J. VAN LEEUWEN, ''Wire-routing is NP-complete,'' Report No. RUU-CS-82-4, Department of Computer Science, University of Utrecht, Utrecht, the Netherlands, 1982.
14. E. S. KUH, T. KASHIWABARA, AND T. FUJISAWA, On optimum single row routing, *IEEE Trans. Circuits and Systems* **CAS-26** (1979), 361-368.

15. A. S. LaPaugh, A polynomial time algorithm for optimal routing around a rectangle, *in* ''Proceedings 21st Ann. Symp. on Foundations of Computer Science,'' pp. 282-293, IEEE Computer Society, Los Angeles, 1980.
16. A. S. LaPaugh, ''Algorithms for Integrated Circuit Layout: An Analytic Approach,'' Report No. MIT-LCS-TR-248, Doctoral Dissertation, Department of Electrical Engineering and Computer Science, Massachusetts Institute of Technology, Cambridge, Mass., 1980.
17. A. S. LaPaugh and C. H. Papadimitriou, The even path problem for graphs and digraphs, *Networks*, to appear.
18. C. E. Leiserson and R. Y. Pinter, Optimal placement for river routing, *in* H. T. Kung, R. Sproull, and G. Steele (eds.), ''VLSI Systems and computations,'' pp. 126-142, Computer Science Press, Rockville, Maryland, 1981.
19. A. Lingas, Minimum perimeter partitions of planar figures, manuscript (1982).
20. A. Mansfield, Determining the thickness of graphs is NP-hard, *Math. Proc. Cambridge Phil. Soc.*, to appear.
21. A. Mansfield, ''Topics in Computational Complexity,'' Doctoral Thesis, University of Oxford, England, 1982.
22. N. Megiddo and A. Tamir, ''On the complexity of point covering and line covering,'' Discussion Paper No. 475, Kellogg Graduate School of Management, Northwestern University, Evanston, Ill., 1981.
23. S. C. Ntafos and S. L. Hakimi, On path cover problems in digraphs and applications to program testing, *IEEE Trans. Software Engrg.* **SE-5** (1979), 520-529.
24. S. C. Ntafos and S. L. Hakimi, On structured digraphs and program testing, *IEEE Trans. Computers* **C-30** (1981), 67-77.
25. R. Y. Pinter, Optimal routing in rectilinear channels, *in* H. T. Kung, R. Sproull, and G. Steele (eds.), ''VLSI Systems and computations,'' pp. 160-177, Computer Science Press, Rockville, Maryland, 1981.
26. R. Y. Pinter, Optimal layer assignment for interconnect, *in* ''Proc. 1982 IEEE Int. Conf. on Circuits and Computers,'' Institute of Electrical and Electronics Engineers, New York, 1982.
27. R. Y. Pinter, private communication (1982).
28. R. Raghaven, J. P. Cohoon, and S. Sahni, ''Manhattan and rectilinear wiring,'' Report No. 81-5, Computer Science Department, University of Minnesota, Minneapolis, Minn., 1981.
29. R. Raghaven and S. Sahni, ''Single row routing,'' Report No. 80-22, Computer Science Department, University of Minnesota, Minneapolis, Minn., 1980.
30. R. Raghaven and S. Sahni, ''The complexity of single row routing,'' Report No. 81-31, Computer Science Department, University of Minnesota, Minneapolis, Minn., 1981.
31. D. Richards, Complexity of single-layer routing, manuscript (1981).
32. R. L. Rivest, A. E. Baratz, and G. Miller, Provably good channel routing algorithms, *in* H. T. Kung, R. Sproull, and G. Steele (eds.), ''VLSI Systems and computations,'' pp. 153-159, Computer Science Press, Rockville, Maryland, 1981.
33. H. Röck, Bus routing problems on acyclic networks, in J. R. Mühlbacher (ed.), ''Proc. 7th Conf. on Graph Theoretic Concepts in Computer Science,'' pp. 99-113, C. Hanser Verlag, Vienna, 1982.
34. A. Siegel and D. Dolev, The separation for general single-layer wiring barriers, *in* H. T. Kung, R. Sproull, and G. Steele (eds.), ''VLSI Systems and computations,'' pp. 143-152, Computer Science Press, Rockville, Maryland, 1981.
35. H. C. So, Some theoretical results on the routing of multilayer printed wiring boards, *in* ''Proc. IEEE Symp. on Circuits and Systems,'' pp. 296-303, Institute of Electrical and Electronics Engineers, New York, 1974.
36. T. G. Szymanski, Dogleg channel routing is NP-complete, manuscript (1982).
37. T. G. Szymanski and M. Yannakakis, private communication (1982).

38. B. S. TING AND E. S. KUH, An approach to the routing of multilayer circuit boards, *in* ''Proc. IEEE Symp. on Circuits and Systems,'' pp. 902-911, Institute of Electrical and Electronics Engineers, New York, 1978.
39. B. S. TING, E. S. KUH, AND A. SANGIOVANNI-VINCENTELLI, Via assignment problem in multilayer printed circuit board, *IEEE Trans. Circuits and Systems* **CAS-26** (1979), 261-271.
40. M. TOMPA, An optimal solution to the wire-routing problem, *in* ''Proceedings 12th Ann. ACM Symp. on Theory of Computing,'' pp. 161-176, Association for Computing Machinery, New York, 1980.
41. S. TSUKIYAMA, I. SHIRAKAWA, AND S. ASAHARA, An algorithm for the via assignment problem in multilayer backboard wiring, *IEEE Trans. Circuits and Systems* **CAS-26** (1979), 369-377.
42. S. TSUKIYAMA AND E. S. KUH, Double-row planar routing and permutation layout, *Networks* **12** (1982), 287-316.
43. P. VAN EMDE BOAS, Another NP-complete covering problem, manuscript (1982).
44. O. VORNBERGER, ''Komplexität von Wegproblemen in Graphen,'' Doctoral Thesis, Fachbereich Matematik/Informatik, Der Gesamthochschule Paderborn, West Germany, 1980.
45. J. A. WALD AND C. J. COLBOURN, Steiner trees in outerplanar graphs, *in* ''Proceedings 13th Southeastern Conference on Combinatorics, Graph Theory, and Computing,'' Utilitas Mathematica Publishing, Winnipeg, Ont., 1982 (to appear).
46. J. A. WALD AND C. J. COLBOURN, Steiner trees, partial 2-trees, and minimum IFI networks, manuscript (1982).