The Solution to Berlekamp's Switching Game*

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ABSTRACT

Berlekamp's game consists of a 10×10 array of light-bulbs, with 100 switches at the back, one for each bulb, and 20 switches at the front that can complement any row or column of bulbs. For any initial set S of bulbs turned on using the back switches, let $f(S)$ be the minimal number of lights that can be achieved by throwing any combination of row and column switches. The problem is to find the maximum of $f(S)$ over all choices of S. We show that the answer is 34. We also determine the solution for $n \times n$ arrays with $1 \leq n \leq 9$.

* This paper appeared in ''Discrete Math.'', vol. 74 (1989), 263–290.

1. Introduction

Several recent papers have studied the covering radius of codes ([1]-[15]). Although a number of constructions for codes with low covering radius are now known, it seems fair to say that the general principles which ensure that a code has low covering radius are not at all well understood.

In order for a binary linear code of length N to have covering radius R, for every binary N-tuple x there must be a codeword within Hamming distance R of x (and furthermore some x must be at exactly Hamming distance R from the closest codeword). In other words the codewords must efficiently ‘‘cover’’ the space of all binary N-tuples. Equivalently, a code C has covering radius R if, given any N-tuple x, it is possible to reduce the Hamming weight of x to at most R by adding to x a sequence of generating codewords for C.

One obvious construction is to take N to be a composite number, say $N = mn$, so that codewords can be represented by $m \times n$ rectangular arrays of 0's and 1's, and to take as generating codewords all single rows and columns of the array. This construction at least has the appearance of distributing the codewords uniformly over the space.

The resulting ‘‘light-bulb’’ codes (the name is explained below) have been the subject of several investigations ([2], [6], [7], [11], [12]). When $N = n^2$ is a perfect square, it is known that this code (of length n^2 and dimension $2n - 1$) has covering radius R_n satisfying

$$\frac{n^2}{2} - \frac{n^{3/2}}{2} + o(n^{3/2}) \le R_n \le \frac{n^2}{2} - \frac{n^{3/2}}{\sqrt{2\pi}} + o(n^{3/2}) .$$

As to the exact values of R_n, up to now it was only known that $R_1 = 0$, $R_2 = 1$, $R_3 = 2$, $R_4 = 4$, $R_5 = 7$, $R_7 \le 16$, $22 \le R_8 \le 23$, $R_9 \le 29$ and $32 \le R_{10} \le 37$ ([2],[7]).

The case $n = 10$ is of particular interest because of the existence at Bell Labs in Murray Hill of a game built by Elwyn Berlekamp some twenty years ago (see Plate 1). There are 100 light-bulbs, arranged in a 10×10 array. At the back of the box there are 100 switches, one for each bulb. On the front there are 20 switches, one for each row and column. Throwing one of the rear switches changes the state of a single bulb, while throwing one of the front switches complements a whole row or column of bulbs.

For any initial set S of bulbs turned on using the rear switches, let $f(S)$ be the minimal number of lights that can be achieved by throwing any combination of row and column switches. The problem, up to now unsolved, is to determine the maximum of $f(S)$ over all choices of S.

This problem is exactly the equivalent to determining the covering radius R_{10} of the 10×10 light-bulb code defined above, and justifies our name for these codes. (There are 2^{19} codewords of length 100 in Berlekamp's game, namely the arrays of lights that can be reached from the all-dark state by using row and column switches only.)

In this paper we show that the solution to this game is $R_{10} = 34$ (a set of 34 lights that cannot be reduced may be seen in Plate 1), and we also determine R_n for $n \le 9$.

The results are summarized in Table I. For each n the table gives the length $N = n^2$

and dimension $K = 2n - 1$ of the light-bulb code, the covering radius R_n, and for comparison the known bounds on $t[N, K]$, the smallest possible covering radius of *any* $[N, K]$ code. For $n \leq 8$ these bounds are taken from [7]. For $4 \leq n \leq 10$ the lower bound on $t\,[N, K]$ is the sphere bound [7, Eq. (56)]. For $n = 9$ and 10 no better codes than these light-bulb codes are presently known.

Figure 1 given examples of extremal sets S of lights corresponding to the entries R_n in Table I.

Light-bulb codes are also of interest because certain optimal covering codes have a very similar structure (see [7, Eqs. (46), (47)]). See also [12].

Section 2 describes our notation and establishes that the values in Table I are lower bounds on R_n. The remaining sections prove in turn that $R_6 < 12$, $R_8 < 23$, $R_9 < 28$ and $R_{10} < 35$. ($R_7 < 17$ is established in [7, Eq. (81)].) The proofs for $n \leq 9$ were done ‘‘by hand’’, and can be checked by the reader without recourse to a computer. Unfortunately many sections of the proof of the upper bound on R_{10} required extensive use of computers, and here we do not attempt to give the complete proof.

2. Notation and lower bounds

As is customary we describe $n \times n$ arrays of light-bulbs both in $\{0,1\}$-notation (0 = off, 1 = on) and $\{+1, -1\}$-notation ($+1$ = off, -1 = on). In ± 1 notation the quantity we wish to determine is given by the formula

$$R_n = \frac{n^2}{2} - \frac{1}{2} \min_{\alpha} \max_{\beta, \gamma} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_i \alpha_{ij} \gamma_j , \tag{1}$$

where $\alpha_{ij}, \beta_i, \gamma_j \varepsilon \{+1, -1\}$. Here $\alpha = \{\alpha_{ij}\}$ is the array of lights, $\beta = \{\beta_i\}$ specifies the settings of the row switches, and $\gamma = \{\gamma_j\}$ the column switches. The same array of lights is described by the $\{0,1\}$-matrix

$$A = \frac{1}{2} (J - \alpha) ,$$

where J is an $n \times n$ matrix of 1's. Given α and β, we maximize $\sum \beta_i \alpha_{ij} \gamma_j$ by setting $\gamma_j = 1$ if $\sum_i \alpha_{ij} \beta_i \geq 0$ and otherwise setting $\gamma_j = -1$. Therefore

$$R_n = \frac{n^2}{2} - \frac{1}{2} \min_{\alpha} \max_{\beta} \sum_{j=1}^{n} | \sum_{i=1}^{n} \alpha_{ij} \beta_i | . \tag{2}$$

If we define

$$C(\alpha , \beta) = \sum_{j=1}^{n} | \sum_{i=1}^{n} \alpha_{ij} \beta_i | ,$$

$$t_n = \min_{\alpha} \max_{\beta} C(\alpha , \beta) ,$$

then

$$R_n = \frac{n^2}{2} - \frac{1}{2} t_n . \tag{3}$$

We focus on the determination of t_n. The lower bounds on R_n are based on:

Lemma 1. *The following three assertions are equivalent. (a)* $R_n \geq p$; *(b)* $t_n \leq n^2 - 2p$; *(c) There is an* α *matrix with exactly* p -1*'s and no more than* $n/2$ -1*'s in each column such that* $C(\alpha , \beta) \leq n^2 - 2p$ *for every* β *in which no more* *than half of the* $\beta_i = -1$ *(i.e.* $\sum \beta_i \geq 0$*).*

Proof. The equivalence of (a) and (b) is immediate from (3). Suppose (c) holds. Then, since the β not considered in its statement are the negatives of those specified, and $C(\alpha, -\beta) = C(\alpha, \beta)$, it follows that $\max_\beta C(\alpha, \beta) \leq n^2 - 2p$ and $t_n \leq n^2 - 2p$. Hence (c)   (b). Suppose (b) holds with $R_n = q \geq p$. Then there must be an α matrix with q -1's and no more than $n/2$ -1's in each column such that $C(\alpha, \beta) \leq n^2 - 2q$ for all β for which $\sum \beta_i \geq 0$. If $q - p$ of the -1's in α are changed to $+1$ to yield α', then $C(\alpha', \beta) \leq n^2 - 2q + 2(q-p) = n^2 - 2p$ for all such β, and therefore (b)   (c).

Remark. When n is even, only half of the β in (c) that have exactly $n/2$ -1's need to be checked for $C(\alpha, \beta) \leq n^2 - 2p$ so long as none of these is the negative of another.

Theorem 2. $R_6 \geq 11$, $R_7 \geq 16$, $R_8 \geq 22$, $R_9 \geq 27$, $R_{10} \geq 34$.

Proof. We claim that the α matrices corresponding to the arrays in Fig. 1 satisfy condition (c) of Lemma 7, and establish the lower bounds.

For example, to show that $R_6 \geq 11$, let α correspond to the 6×6 $\{0,1\}$-matrix A in Fig. 1. Then $C(\alpha, \beta) = 14 = 6^2 - 2 \cdot 11$ when $\beta_i = 1$ for all i. The corresponding absolute values of the column sums, $|\sum_i \alpha_{ij} \beta_i|$ are 642200. We refer to a β with one $\beta_i = -1$ as a *single*, a β with two $\beta_i = -1$ as a *double*, and a β with three $\beta_i = -1$ as a *triple*.

Singles. Since there is at most one -1 in a row preceding the final two columns, a single cannot give $C(\alpha, \beta) > 14$. For example, the column absolute sums with $\beta_1 = -1$ are 460022.

Doubles. When two rows are reversed, the column absolute sums include a 6 only when the double is 23 ($C = 14$) or 45 ($C = 10$). Otherwise, columns 1, 3 and 4 contribute a total of $2 + 2 + 2 = 6$ to C. The maximum obtainable from columns 2, 5 and 6 for C is 8 (doubles 12, 14, 15, 16, 24, 56), so $C \leq 14$ for all doubles.

Triples. We consider only triples that contain row 1 since the others are complements of these. Three triples have $C = 14$, namely 123 (column absolute sums 024422), 145 (sums 024422) and 156 (sums 024062). In all other triples except 124 ($C = 10$), the contribution to C from columns 1, 2, 5 and 6 is $0 + 2 + 2 + 2 = 6$ and at most 4 from columns 3 and 4.

We have now dealt with all the necessary β's, and so $R_6 \geq 11$ follows from Lemma 1(c).

We have obtained similar proofs (‘‘by hand’’) for the other lower bounds. However, these proofs may also be - and were - carried out trivially by computer (even for $n = 10$ there are only 512 β's to consider). For this type of argument some readers will prefer the argument ‘‘by hand’’, but others may find the computer verification more convincing. For this theorem there is no difficulty in carrying out either form of proof, and so we omit the remaining details.

3. Upper bounds, and the case n = 6

The upper-bound proofs also use Lemma 1(c), as the basis for proof by contradiction, but these proofs are more involved since all α matrices that satisfy the initial conditions of (c) must be shown to have $C(\alpha, \beta) > n^2 - 2p$ for some β with $\sum \beta_i \geq 0$. We organize these proofs around the distributions of column sums of the α matrices, ordered left to right by decreasing magnitudes, whose members add to $n^2 - 2p$. These distributions are then considered sequentially and eliminated in turn on the basis of $C(\alpha, \beta) > n^2 - 2p$ for some β. We say that a distribution is *out* when it has been shown that, for every α adhering to the initial conditions of (c) with the noted column sums, $C(\alpha, \beta) > n^2 - 2p$ for some β.

To illustrate, suppose we wish to show that $R_6 < 12$, i.e. $R_6 \geq 12$ is impossible. With $p = 12$, $n^2 - 2p = 12$. With at least as many +'s as −'s in each column of α, there are seven column-sum distributions whose members add to 12, namely 660000, 642000, 622200, 444000, 442200, 422220 and 222222. The first of these has no −'s in the first two columns and three −'s in each of the last four columns. The distribution 222222 has four +'s and two −'s in each column.

Some of the distributions are easy to get out. For example, one $\beta_i = -1$ for 660000 yields new absolute column sums 442222, and the sum of these exceeds 12, so 660000 is out. Other distributions are more difficult to get out, and before giving the full proof we note some general principles that are used in the upper-bound proofs.

Assume in the statements of the following principles that n is even, $n \geq 6$, α is an

$n \times n$ ± 1 matrix with p -1's and at least as many +'s as −'s in each column, and $C_0 = n^2 - 2p$ (the sum of the column sums with all $\beta_i = 1$). Throughout this paper we use d_k to denote the number of columns with column absolute sum k, for $k = 0, 1, 2, \ldots$.

P1. *If some row has exactly x −'s in columns with positive sums and $x + d_0 > n/2$, then $C > C_0$ when that row is reversed.*

Proof. Given the hypotheses, the row reversed adds $2(x + d_0)$ to C_0 and subtracts $2(n - x - d_0)$ from C_0, for a net increase of $4(x + d_0) - 2n > 0$.

P2. *If $\lceil d_0/2 + p/n \rceil > n/2$, then the reversal of some row gives a $C > C_0$.*

Proof. The d_0 columns use $d_0\, n/2$ −'s, leaving $p - d_0\, n/2$ for the other columns. Hence one of the rows has at least $\lceil (p - d_0\, n/2)/n \rceil = \lceil p/n - d_0/2 \rceil$ −'s in the columns with positive sums. If this number plus d_0 exceeds $n/2$, then P1 says a reversal of that row gives a $C > C_0$.

P3. *If $d_0 = 0$ and either $d_2 = n$ or $\{d_2 = n-1,\ d_n = d_{n-2} = 0\}$, then the reversal of some two rows gives a $C > C_0$.*

Proof. If $d_2 = n$, reverse two rows that have −'s in some column. This increases that column's sum to 6 and since all other column sums stay at 2 or increase to 6, $C > C_0$. If $d_2 = n-1$ and $d_n = d_{n-2} = d_0 = 0$, reverse two rows that have −'s in the column whose sum exceeds 2. That column's sum increases by 4 and the others stay at 2 or increase to 6, so $C > C_0$.

Theorem 3. $R_6 < 12$.

Proof. We suppose $R_6 \geq 12$ and obtain a contradiction using the method described at the beginning of this section. In the present case $p = 12$ and $C_0 = 6^2 - 2(12) = 12$. Since P2 shows that a column sums distribution is out if $d_0 \geq 3$, and P3 gets 222222 out, we are left with the distributions (#1) 622200, (#2) 442200 and (#3) 422220 for further consideration. By P1, the only way that #1 can avoid going out is to have

6	2	2	2	0	0
+	−	+	+		
+	−	+	+		
+	+	−	+		
+	+	−	+		
+	+	+	−		
+	+	+	−		

Given this pattern in the d_2 columns and the fact that there must be two rows with the same pattern in the d_0 columns (six rows but only four possible patterns under 00, namely $++$, $+-$, $-+$ and $--$), reversal of two such rows gives column absolute sums at least as great as 222244, whose sum exceeds 12. Hence #1 is out.

When two rows with $-$'s in a d_2 column of #3 are reversed, its new sums are either 062220, which is out since #1 is out, or the sum of its column sums exceeds 12. Hence #3 is out.

Finally, from P1, the only way that #2 can avoid going out is to have

4	4	2	2	0	0
−	+	+	+		
+	−	+	+		
+	+	−	+		
+	+	−	+		
+	+	+	−		
+	+	+	−		

Switch rows 1, 3 and 4 to get column absolute sums at least as great as 224422, whose sum exceeds 12. Hence #2 is out.

Since all possible distributions of column sums are out, we contradict $R_6 \geq 12$.

4. The case n = 8

Theorem 4. $R_8 < 23$.

Proof. We suppose that $R_8 \geq 23$, so $p = 23$ and $C(\alpha, \beta) = n^2 - 2p = 18$ for the application of Lemma 1. There are 17 distributions of columns sums for α that adhere to Lemma 1(c) whose members add to 18. Of these, seven are out by P2 (their distributions have four or more 0's, i.e. $d_0 \geq 4$) and one (42222222) is out by P3. The other nine are (#1) 82222200, (#2) 64422000, (#3) 44222220, (#4) 64222200, (#5) 62222220, (#6) 44442000, (#7) 84222000, (#8) 66222000, (#9) 44422200, listed in the order in which we now eliminate them.

In the proofs of Theorems 4-6 we shall describe arrays of lights by 0's (rather than +'s) and 1's (rather than −'s), although column sums will still be specified in the ±-notation. A column sum of s indicates a column with $(n-s)/2$ 1's and $(n+s)/2$ 0's.

Since #1 has 23–8=15 1's in columns 2 through 6 and P1 forces at most two 1's in a row prior to the d_0 columns (else #1 is out), one row has exactly one 1 in the d_2 columns and the other seven rows each have two 1's in these columns. If we have a block of four 1's $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ in the d_2 columns then reversal of the two rows gives column sums at least 46622200, which add to 22 and get #1 out, or if we have

2	2	2
1	1	0
1	0	1
0	1	1

then switching these three rows gives column sums at least 24444422, which add to 26 and get #1 out. In view of these facts, #1 can avoid going out only if it is constructed (top down, with row 1 having the single 1 under d_2) as in Fig. 2a. However, completion of the last two d_2 columns forces one of the out patterns noted above, so #1 is out in any case.

When the row in #2 with the 1 in the first column is switched, its column sums are at least 82200222, which is out. Hence #2 is out.

When two rows with 1's in column one of #3 are reversed, its sums are at least 80222220, which is #1. Hence #3 is out. Similarly, when two rows in #4 are reversed (those with the 1's in d_4), we get at least 28222200, so #4 is out.

For #5, reverse the row with the 1 in the d_6 column and another row that agrees with it in the last column to get columns sums at least 62222224. These add to 22, so #5 is out.

For #6, switch three rows that in two of the d_4 columns have 10, 10, 01 (or more 1's) to get at least 62220222, which is #5 and out.

When the two rows with 1's in the d_4 column are reversed in #7 along with a third row, the column absolute sums are at least 26000222 (sum of 14). If one of these 0's can be made a 4 by choice of the third row to reverse, which can be done if one of the d_2 columns has a 1 in the first two rows chosen for reversal, then we get #4, which is out. Otherwise (no 1's in the d_2 columns in the two rows that have 1's under d_4) the third row can be chosen to have at least two 0's in the three d_2 columns, which transforms 000 in 26000222 into at least 440 so #7 is out.

When a row with a 1 in the d_6 column in #8 is reversed, its sums are at least 84000222, which is #7 and out.

Finally, if some row for #9 has two 1's in the d_4 columns, reversal of this and another row gives at least 84022200, which is #7 and out. Otherwise, the pattern is as shown in Fig. 2b for the first three columns. If one of the d_2 columns has two 1's in the first six rows, reversal of these two gives at least 80062200 or 44062200, both of which are out. Otherwise the d_2 have solid 1's in the last two rows, as shown in Fig. 2b, and reversal of these two gives at least 00066600, which is out by P1. This completes the proof.

5. The case n = 9

Theorem 5. $R_9 < 28$.

Proof. We suppose that $R_9 \geq 28$ and obtain a contradiction. With $p = 28$,

$n^2 - 2p = 25$. Since n is odd, the column sums are $1, 3, 5$,... instead of $0, 2, 4$,... . The distribution of column sums for α matrices that have a total of p −'s, more +'s than −'s in each column, and whose members add to 25, are given in Table II. These are listed lexicographically but numbered by their order of elimination.

#1. Since $d_1 = 7$, all 28 1's are in the last seven columns, so some row has at least four 1's. Switch that row to get at least 773333111 with sum 29. Since this exceeds $25 = n^2 - 2p$, #1 is out.

#2. The 27 1's in the last seven columns must be distributed three to a row, else #2 goes out like #1. But then the row with the 1 under d_7 has three other 1's, and switching this row gives sum at least 29, out.

#3. If a row has four 1's in the d_1 columns, switching such a row gives column sums whose sum exceeds 25. Otherwise, at least six rows have exactly three 1's under the d_1, and either one of these also has a 1 under a d_5 (reversal gives 773333111 and out) or the 1's under the d_5 are confined to two or three rows and reversal of a row with two 1's under the d_5 gives 777111111 or more.

#4. If a row has four 1's in the six d_1 columns, #4 is out like #3. Assume otherwise, so at least six rows have three 1's under the d_1. If such a row also has a 1 under a d_7 or d_5, #4 goes out by one reversal. Otherwise the four 1's under d_5 and the two d_7 are in three rows, and either the 1's under the d_7 are in the same row (993111111 by reversal) or some row has a 1 under d_5 and a d_7 (957111111 by reversal).

#5. If a row has 1's under d_5 and a d_3, reversal gives at least 775111111, which is #4 and out. If two rows have four 1's under the d_3, reversal of these two gives at least 517711111, which is #4 and out. Assume henceforth that the rows with a 1 under d_5 have no 1 under a d_3, and that the 1's in the two d_3 columns involve either five or six rows. To avoid going out, each row with a 1 under d_5 or d_3 can have at most two 1's under the d_1, and if one row has two 1's under the d_3 it must have no 1's under the d_1. Assume these restrictions on the d_1. There are 20 1's in the five d_1 columns. If the 1's under d_5 and the d_3 involve exactly seven rows (one of which has two 1's under the d_3), then the other rows have at least eight 1's under the d_1. If one of these two has five 1's, its reversal puts #5 out, so we may assume that each has exactly four 1's under the d_1. Alternatively, if d_5 and the d_3 involve eight rows, the other row has at least four 1's under the d_1, and #5 is out by reversal if it has five 1's. Consequently we may suppose in either case that one row has four 1's under the d_1 *and* that the two rows with a 1 under d_5 each have exactly two 1's under the d_1. When all three rows are reversed, we get at least 373333311, so #5 is out.

#6. Since a row with 1's in the two d_7 columns gives #1 or more by reversal, and a row with 1's under a d_7 and a d_3 gives #3 or more by reversal, assume that a row with a 1 under d_7 has no other 1's prior to the d_1. Also, no row with a 1 under d_7 can have more than two 1's under the d_1, and similarly for d_3, and no row with two 1's under d_3 can have any 1's under the d_1, else one row reversal gets #6 out. And, as in #5, we can assume that a row with no 1's under the d_7 and d_3 has at most four 1's under the d_1. If three rows have all the 1's under the d_3, then reversal of these three gives #1 and more; if two rows have two 1's each under the d_3, then reversal of these two along with a row having a 1 under d_7 yields 515511333 and if either one or no row has two 1's under the

d_3, then the row(s) with no 1's under the d_7 and d_3 have exactly four 1's each and the rows with a 1 under d_7 have exactly two 1's each under the d_1, and when these latter two rows are reversed along with a row with four 1's under the d_1 we get 553333311 or more. Hence #6 is out.

#7. If a row has 1's under d_7 and a d_5, or under both d_5, or under d_7 and d_3, then #7 is out by a single reversal (#2 or #4 or #5). Moreover, the usual restrictions on the number of 1's under the d_1 imply that if some row has 1's under d_3 and a d_5, then reversal of this row and the row with 1 under d_7 gives at least 751333311, with sum 27. Hence, to avoid going out, eight rows each have one 1 in the first four columns and two 1's under the d_1, and the ninth row has four 1's under the d_1. Then some two of the five rows with a 1 under d_7 or the d_5 must have 1's in the same d_1 column, and when these two are reversed we get 751151133, 391151133 or 355151133, each of which sums to 27.

#8. If a row has three or four 1's under the d_3, reverse this row to get #8 out. Otherwise, at least three rows each have two 1's under the d_3, and each such row must have all 0's under d_1 to avoid going out by a single reversal (with one 1 we get 755113111, which is #7 and out). When three such rows are reversed we get at least 391115555 or 355115555, so #8 is out.

#9. Since reversal of a row with two 1's under the d_5 gives at least 773311111, which is #6 and out, assume that each of eight rows has one 1 under the d_5 and exactly two 1's under the d_1 (three of the latter get #9 out), with exactly four 1's under the d_1 in the ninth row. If any two of the eight rows have their 1's under the d_1 in only two columns, then reversal of the two rows gets #9 out with a sum of 31, so assume otherwise for these rows. Then if two rows with 1's under one d_5 have two 1's in the same d_1 column, reversals of these two along with a third row with a 1 in the same d_1 column and a 1 under another d_5 column give 731171115, with sum 27, so assume this case is also avoided. Finally, switch four rows consisting of the ninth row (four 1's under the d_1) and three other rows that have 0's in the same d_1 column that has a 0 in the ninth row. The column sums for the d_5 columns are either 5331 or 3111, and those for the d_1 are 97111 or 75511, so #9 is out since the total sum either exceeds 25 or else is exactly 25 with #2 or #7.

#10. If a row has three or more 1's in the first five columns, #10 goes out when that row is reversed, so assume no row has more than two 1's in the first five columns. Then, since there are 12 1's in the first five columns, at least three rows have two 1's each in those columns. In addition, if a row has 1's under d_7 and d_5, or d_7 and d_3, or d_5 and d_3, then it must have all 0's under the d_1 since one 1 there would yield #2, or #5, or #7 by a reversal. If there are two such rows (e.g. 101000000 and 010100000) then switching these two give column sums that exceed 25 (753313333 as illustrated). If there is one row with at least one of its two 1's in the first five columns under d_7 or d_5, and a second row that has two 1's under the d_3, then reversal of these two gives column sums that exceed 25 (the least is 353331333) since the latter row (d_3) can have at most one 1 under the d_1. Hence the only way not to go out is with exactly three rows with two 1's under the d_3 as in Fig. 3a. Reversal of these three gives #9.

#11. Assume as in #10 that no row has more than two 1's in the first five columns. If such a row has 1's under the d_5 or a 1 under a d_5 and another under a d_3, then it must have 0's under all d_1 or else it goes out by #6 or #10. If there are two rows like this, or one like this and another with two 1's under the d_3, then as in #10 reversal of these two gives column sums that exceed 25. Since at least three rows have two 1's in the first five columns, #11 is out unless each of these has its two 1's under the d_3. Reversal of these three gives 111991115 or more.

#12. Assume that no row has more than two 1's under the d_3, or else we get at least 555511111 (#9). Let row 1 be the row with a 1 under d_7. If row 1 has two 1's under the d_3, reversal gives #3 or more, hence out. Suppose row 1 has exactly one 1 under the d_3, in column 2. Reverse row 1 and the other two rows with 1's in column 2 to get at least 5911331111, which is #5. Suppose finally that row 1 has no 1's under the d_3. Since there are 15 1's under the d_3 and no row can have more than two of these, seven of the last eight rows each has two 1's under the d_3 and the other has one 1 under the d_3. In addition, each of the seven has at most one 1 under the d_1 else reversal gives 555111331, which is #11. Consequently, since there are 12 1's under the d_1, at least five of these must be in row 1 and the row with only one 1 under the d_3. Reversal of these two gives 731111551, which is #7.

#13. Assume that no row has more than two 1's in the first six columns, else reversal gives #4, #7, #11, or more. Also assume than no row has two 1's under the d_5 since otherwise reversals give at least #1 or #6. Then, if the four rows with a 1 under the d_5 have a total of two or fewer 1's under the d_3, reversal of all four gives at least 551155111 (#9). If the same four rows each has a 1 under the d_3, then each of these has at most one 1 under the d_1 (else 735111331, which is #10). In this case, if two rows with 1's under one d_5 have four 0's in two d_1 columns, reversal of the two gives at least 913311331 (#8), and to avoid going out we need the arrangement in Fig. 3b, with a repeat, under permutation, of rows 1 and 2 under the d_1 in rows 3 and 4. (If the 1's under the d_3 align with at least two in one column, reversal of two rows for those 1's gives either #2 or #7.) If the double 0 in a d_1 column for rows 3 and 4 is in column 7, then reversal of the first three rows gives #10, and if this double zero is in column 8 or 9 then reversal of rows 3, 4 and one of the first two rows again gives #10.

Hence, to avoid going out, we assume that exactly three of the four rows with a 1 under the d_5 also have a 1 under the d_3 (in different columns) and at most one 1 under the d_1 (else get #10). By the analysis of the preceding paragraph, we can assume we have the array in Fig. 3b with the 1 in row 4 and column 6 changed to 0. Since reversal of rows 1 and 3 gives 553131..., and #11 is out, there must be a 0 in row 3, column 8; similarly, reversal of rows 2 and 3 forces a 0 in row 3, column 9. Reversal of rows 1, 2 and 4 then requires a 1 in row 4, column 7 (else get #10) and, similarly, reversals of 1, 3 and 4 and then 2, 3 and 4 force 1's in row 4, columns 9 and 8, respectively. Thus we have Fig. 3c. Finally, switching rows 1, 2 and 4 gives #12.

#14. Assume that no row has more than three 1's in the first seven columns, else a reversal gives column sums exceeding 25. Let rows 1 and 2 have the 1's under d_5. Suppose first that some d_3 column has 1's in rows 1 and 2. Then reversal of rows 1 and 2 gives 97..., which is out (#2) unless rows 1 and 2 have solid 0's under the other d_3 rows; but then reversal of rows 1, 2 and a third row with a 1 in the special d_3 column gives 791133311 or more. Suppose henceforth that no d_3 column has 1's in rows 1 and

2.

Suppose next that row 1 has 1's in columns 2 and 3 (first two d_3), so it must have 0's in all remaining columns, including the d_1 (else we get #7). Then row 2 has at most one 1 under the d_3, else reversal of rows 1 and 2 gives at least 933331111 (#8). If row 3 has 1's under columns 2 and 3, then reversal of rows 1 and 3 gives at least 5771...1 (#4), so assume that the other four 1's for columns 2 and 3 are in rows 3, 4, 5 and 6. If any one of these four rows has more than one other 1 under the d_3, reversal of it and row 1 gives at least 573331111 (#10), so assume that each of rows 3 through 6 has no more than one 1 in columns 4 through 7. This then forces the final three rows to have at least seven 1's in columns 4 through 7, and reversal of the last three rows gives at least 133555111 (#11) or 133951111 (#5).

Since a similar result obtains if row 2 has two 1's under the d_3, assume henceforth that each of rows 1 and 2 has at most one 1 under the d_3. Suppose next that row 1 has a 1 in column 2 and row 2 has a 1 in column 3. If row 4 has 1's in columns 2 and 3, then reversal of the first three rows gives at least 7551333111, so assume that the other four 1's in columns 2 and 3 are in rows 3, 4, 5 and 6. If one of these has fewer than two 1's under the other d_3, then reversal of that row along with rows 1 and 2 gives at least 751133311 (#10), so assume each of rows 3 through 6 has exactly two 1's in columns 4 through 7. Then each of rows 3 through 6 must have 0's in the last two columns, else a single reversal gives at least 351551113 (#11). In addition, rows 1 and 2 can have at most one 1 in column 8 and one 1 in column 9, else reversal of rows 1 and 2 gives at least 933111115 (#5). Consequently the last three rows have solid 1's in columns 8 and 9, and reversal of these three gives at least 133111177 (#6).

We conclude that #14 is out unless it has at most one 1 under the d_3. Suppose it has exactly one such 1, say in row 1, column 2. Let row 3 also have a 1 in column 2. Then reversal of rows 1 and 3 gives at least 751133311 (#10), so assume henceforth that there are no 1's in rows 1 and 2 under the d_3. Then there are 18 1's under the d_3 in the last seven rows, so at least four of these rows must have three 1's under the d_3. Each such row must have 0's in the last two columns, else reversal gives #11. Then reversal of these four rows gives at least 333111177, which exceeds 25 in sum.

#15. Assume no row has more than three 1's under the d_3, else reversal gives #9 or more. Since there are 24 1's under the d_3, at least six rows must have three 1's under the d_3. If two of these six rows have 1's in the last column, their reversal gives at least 333333115 (#14) or 733331115 (sum exceeds 25) and so forth, so #15 goes out. Assume therefore that all rows with three 1's under the d_3 collectively have one 1 under d_1, with the other three 1's under d_1 in rows with exactly two 1's under the d_3. The four rows with 1 under d_1 can be arranged as in Fig. 3d (with possible rearrangements inside the parentheses), since if any of rows 2, 3 and 4 has a 1 in the first three columns, reversal of that row and row 1 gives #10. Reversal of rows 2, 3 and 4 in Fig. 3d gives at least #10, so #15 is out. This completes the proof.

6. The case n = 10

Theorem 6. $R_{10} < 35$.

Remark. Our proof of this result makes extensive use of computers. We have a proof of the weaker result that $R_{10} < 36$ which does not require computers, but it is quite long and we do not give it.

Proof. We suppose that $R_{10} \geq 35$, and let α denote a 10×10 ± 1-array containing $p = 35$ -1's satisfying the hypothesis of Lemma 1(c), and in which every column sum is nonnegative. The column sums of α add to $C_0 = 10^2 - 2 \cdot 35 = 30$. Let A denote the corresponding $\{0,1\}$-array, containing 35 1's and with at least as many 0's as 1's in each column. Statements about column sums will always refer to α, but we shall work with A when trying to construct the array. We first record some properties of A.

P4. *(a) d_0 (the number of columns with sum zero) is at most 3. (b) Let x be the number of 1's in any row of A that belong to columns with positive sums. Then $d_0 = 3$ implies $x \leq 2$, $d_0 = 2$ implies $x \leq 3$, $d_0 = 1$ implies $x \leq 4$, and $d_0 = 0$ implies $x \leq 5$.*

Proof. (a) follows from P2 and (b) from P1.

P5. *If $d_0 = 3$ then each row of A contains exactly two 1's in the columns with positive sums.*

Proof. This follows from P4, since there must be a total of 20 1's in these columns.

Let u be a row of A. The number of 1's in u belonging to columns with sums > 2 is called the *height* of u, and the entries belonging to columns with sum 0 (i.e. the last d_0 entries) form the *tail* of u.

P6. *If $d_0 = 3$ there are two rows with identical tails.*

Proof. There are ten rows, but only 2^3 possible tails.

P7. *If* $d_0 = 2$ *the height of any row is at most 3.*

Proof. Otherwise reversing that row gives a $C > C_0$ (i.e. reduces the number of 1's in A).

Similarly, by considering pairs of rows, we obtain:

P8. *Suppose there are k columns with sums* > 2*, and* $d_0 = 2$ *with sum zero. (a) If there are two rows whose heights add to more than k, then A is out. (b) If there are two rows whose heights add to k, then their tails must be complementary. (c) If there are two rows whose heights add to* $k - 1$*, then their tails must be distinct.*

P9. *If* $d_2 \geq 6$ *then there is a block of four* 1*'s in the* d_2 *columns.*

Proof. A d_2 column contains four 1's, so six such columns require 24 1's, and therefore one row-say the first-must contain at least three 1's. There is now a unique way to complete the first three d_2 columns while avoiding a block of four 1's, namely $1^4\ 0^6,\ 1\ 0^3\ 1^3\ 0^3,\ 1\ 0^6\ 1^3$. But now there is no way to complete the fourth d_2 column without producing a block of four 1's.

P10. Our main weapon for proving Theorem 6 is a computer program that, given a partially completed array of the shape shown in Fig. 4, attempts to add five more rows. It also takes as input a list of all distributions of column sums that are already out. The shaded Γ-shaped region of the figure indicates the part of the array that is already determined. The X's indicate the new rows; asterisks indicate unspecified entries.

For a given partial array (consisting of the original array and up to five new rows), the program looks at the effect of reversing all possible subsets of the known rows. The program announces that this partial array is out if (a) the new absolute values of the column sums add to more than 30, or if they add to exactly 30 and either (b) the new d_0 exceeds 3 (out by P4a), or (c) if the new distribution of column sums is one that is already out. The program does not attempt to fill in the d_0 columns. Instead it uses the parity of the number of rows reversed to obtain a lower bound on the new absolute values of the sums in a d_0 column. For if i rows are reversed then the new column sum is at least 0 if i is even, or at least 2 if i is odd.

If the program is unable to eliminate a partial array it tries all possibilities for the next row (using P4b to restrict the choices). It descends through the tree of possibilities to depth 5, i.e. tries to add ≤ 5 new rows. The final output describes how the arrays were eliminated, and lists the surviving arrays. Equivalent solutions, however, must be weeded out by hand. This often results in a very large number of solutions, and it is then advisable to proceed cautiously down the tree of possibilities, eliminating duplicates at the earliest possible point.

A reference to ‘‘P10’’ in the following proof indicates conclusions obtained with the help of this program.

Table IV below illustrates the operation of this program in attacking case #40 of the proof, starting with the partial array shown in Fig. 11.

P11. As a last resort we used a program which takes a partial array having the shape of the shaded region in Fig. 4, and considers all possible ways to fill in the remaining entries of the 10×10 array that are consistent with the specified column sums. For each such array it then tries reversing every subset of the rows to see if the new column sums add to more than 30. (Actually only 2^9 subsets need be considered, as noted just below Lemma 1.) The phrase ‘‘by P11’’ indicates when this program was used in the proof. It was often used to eliminate a 10×7 or 10×8 partial array in which all except three or two d_0 columns had been completed (see for example case #10). There are $\left\lceil {9 \atop 4} \right\rfloor$ possibilities to be considered for each d_0 column, and since $\left\lceil {9 \atop 4} \right\rfloor^3 \approx 2 \cdot 10^6$ this is a reasonable task for a computer.

We come now to the main part of the proof. There are 40 distributions of column sums for α that sum to 30 and satisfy $d_0 \le 3$; they are listed in Table III. We shall eliminate them in the following order: 1, 2, 3, 6, 4, 19, 5, 7, 8, 15, 16, 9-14, 17, 18, 20-24, 27, 29, 25, 31, 26, 35, 33, 30, 34, 32, 28, 36, 40, 37, 38, 39.

To assist the reader who wishes to check the proof, we have assigned a grade of difficulty to each case, using the traditional adjectival system used for British rock climbs: *E* = easy, *D* = difficult (but without using a computer), *VS* = very severe (possibly with use of a computer), *HVS* = hard very severe, and *XS* = extremely severe (very extensive use of a computer). As with rock climbs, the grades are approximate, and will probably be reduced as easier solutions are found and techniques improve!

#1 (E). Switch the two rows mentioned in P6. The new column sums are at least 6622222444, with sum $\sum = 34$. Since $34 > 30$, #1 is out.

#2 (D). If the 1's in the columns headed 8 and 4 intersect (which we usually abbreviate to ''if the 8 and the 4 intersect''), then reverse that row to get 8 10 6 0 0 0 0 2 2 2, with sum $\sum = 30$, $d_0 = 4$, which is out by P4. So we may assume the 8 and 4 are disjoint:

	10	8	4	2	2	2	2
1	0	1	0	1	0	0	0
2	0	0	1	0	1	0	0
3	0	0	1	0	0	1	0
4	0	0	1	0	0	0	1
5	0	0	0	1	1	0	0

(4)

and that row 1 is as shown. If row 2 is 0011000... then reversing rows 123 (for all choices of row 3) gives $\sum \geq 34$. So row 2 is as shown. If row 3 is 0010100... then again this is out by switching 123. So row 3, and similarly row 4, are as shown. Without loss of generality row 5 is as shown, and now switching 125 gives 4624444222, $\sum = 34$, out.

#3 (D). The same method applies. We force the two 6's to be disjoint, arriving at a configuration similar to (4) (differing from it only in columns 1-3), and reverse 125 to get $\sum = 34$, out.

#6 (D). Similar to #2 and #3.

#4 (VS). If the 6 meets one of the 4's, we reverse that row to get #6. If two 4's intersect in two or more rows (Fig. 5a), there is no way to complete rows 1-4, by P10. If two 4's intersect in one row (Fig. 5b), there is no way to complete rows 1-6 (P10). Finally, suppose the 6 and the 4's are disjoint (Fig. 5c). If row 2 is 0100100... there is no way to complete rows 1-7 (P10). So we may suppose row 2 is as shown in Fig. 5c. There are two possibilities for row 3, 0010100... or 0010001..., but neither may be completed to row 8 (P10). Thus #4 is out.

#19 (D). Consider the five 1's in columns 1-3. By P8a the total height of any two rows is ≤ 3. Therefore the heights of the individual rows are (a) $2\ 1^3\ 0^6$ or (b) $1^5\ 0^5$. (a) Suppose the first row has tail 00. By P8b, rows 2, 3, 4 have tail 11. Then rows 2, 3 violate P8c. (b) By P8c all of rows 1-5 have distinct tails, which is impossible.

#5 (E). Since there are only 8 1's in the d_2 columns, there is a row with two 1's in the d_4 columns. Reverse this row, obtaining #19.

#7 (VS). If there is a row 110... or 1010..., we switch it and get #1 or #3. If we have Fig. 6a, we switch 134 and get #6. If we have Fig. 6b, there is no way to complete rows 1-6 (P10). Finally, if we have Fig. 6c, there are two choices for row 2, namely 0100100... or 0100010... . In neither case is it possible to complete rows 1-7 (P10).

#8 (VS). The 1's in columns 1-3 must be disjoint (or else we get an earlier case by reversing one row). The fourth column must also be disjoint (P10). So the first four columns are $1\ 0^9$, $0\ 1^2\ 0^7$, $0^3\ 1^2\ 0^5$, $0^5\ 1^3\ 0$. By P10 the next three columns begin $\{110, 001, 000\}$, $\{100, 011, 000\}$ or $\{100, 010, 001\}$, but (by P10 again) in each case it is impossible to complete rows 1-8.

#15 (D). If we have rows 011... and 001..., we switch 1 and 2 to get #19. So the 8 is disjoint from the 2's. Suppose row 1 is 01...00. By P8c the remaining nine tails must be 01, 10 or 11. So the distribution of these three types of tail is 333, 432, 441, 540,..., or 900. From P4, the 24 1's in the d_2 columns, which are restricted to rows 2-9, must be arranged into 5 rows of 3 and 4 rows of 2 (*). Consider three rows with the same tail ab. In these rows if there is a d_2 column with two 1's, we reverse those two rows to get sum 34. So there are at most six 1's in these rows, and hence exactly six, by (*). If there is a fourth row with tail ab, that row has no 1's in the d_2 columns, contradicting (*). So the tail distribution is 333. But then there are only 18 1's in the d_2 columns, contradicting (*) again.

#16 (E). If there are two rows 011..., this is out by P8a. If we have 011..., 010..., and 001..., this is out by P8b, P8c. So the 6 is disjoint from the 4, and columns 1-3 are 0^{10}, $1^2\ 0^8$, $0^2\ 1^3\ 0^5$. This is out by P8c.

#9 (VS). The 8 is disjoint from the 6 and the 4's (or else we switch one row and get #15 or 16), and the 6 is disjoint from the 4's (or else we get #19). Thus we have

8	6	4	4	4	2	2
1	0	0	0	0	1	0
0	1	0	0	0	a	b
0	1	0	0	0	c	d
0	0	1	1	0	0	0

where $abcd = 1010$ (out by switching 123), 1001 or 0101, and in the last two cases there is no way to complete rows 1-9 (P10).

#10 (HVS). We first show, by extensive use of P10, that no two d_4 columns contain a solid block of four 1's, and then that the 8 is disjoint from any 4. At this point we have the partial array shown outside the broken line in Fig. 7. There are now five choices for the d_2 column, one of which is shown. For the other four choices of the d_2 column the array cannot be completed (P10). But for the d_2 column illustrated there is a unique way fill in rows 1-10, as shown in the figure (P10). Now there is no way to complete columns 8-10 (P11).

#11 (D). Any two 6's are disjoint, or else we get #7 by reversing one row. So we may assume columns 1-4 are $1^2\ 0^8$, $0^2\ 1^2\ 0^6$, $0^4\ 1^2\ 0^4$, $0^6\ 1^2\ 0^2$. Then some d_2 column contains three 1's in the top 8 rows, and switching these three rows gives $\sum\ \geq 34$.

#12 (HVS). We first use P10 to show that all pairs of 6's are disjoint, and then that all pairs of 4's are disjoint. There are still many cases, but all are eliminated by P10 except that shown in Fig. 8a. This is finally eliminated by P11.

#13 (HVS). Take the d_2 column to be $1^4\ 0^6$, and consider the location of the other four 1's in rows 1-4. There are 13 cases, one of which is shown in Fig. 8b. The other 12 cases are eventually eliminated using P10. The 13^{th} case has a unique completion to ten rows (P10), as shown in the figure. This is finally eliminated by P11.

#14 (HVS). We first use P10 to force the configuration shown outside the broken line in Fig. 9a. Then there is a unique way (P10) to complete rows 1-10, as shown. This is finally eliminated by P11.

#17 (XS). There cannot be a row $01^3 0$... (out by P4) or 011010... (this reduces to #8). If there is a row 0(110)0... with 1's just in two d_4 columns, we say that these columns are paired. Two columns cannot be paired twice (or we get #6). So there are at most three pairings. The cases of 3, 2, 1, 0 pairings are, with difficulty, eliminated in turn using P10. In no case is it possible to complete all ten rows.

#18 (D). Similar to #19.

#20 (XS). Using P10 and P8, we systematically show that the 8 is disjoint from the 6 and the two 6's are disjoint from each other. Then for each choice of the two d_4 columns we attempt to complete the d_2 columns. In several cases - one is shown in Fig. 9b - it is possible to complete all ten rows, but then P11 shows that there is no way to complete columns 8-10.

#21 (XS). We cannot have a row 1110... (or else this reduces to #3 by switching). There are now two cases: either there is a row 1100..., or the 8 is disjoint from the 4's. We systematically fill in rows and columns using P10, working from the top left downwards, and using P11 whenever all ten rows have been completed. Two such partial arrays (out of many) are illustrated in Fig. 10. P11 shows that none of these may be completed to a 10×10 array.

#22 (VS). We use P8 and P10 to force the first four columns to be disjoint, and then use P10 to eliminate this configuration.

#23 (HVS). With the help of P8, we reduce the possibilities for columns 1-3 to just two, the transposes of

11000...		110000...
10100...	or	001100...
00011...		000011...

We now consider all possibilities for column 4, and eliminate each in turn using P10.

#24 (XS). Similar to #23, except that there are more subcases. None survive to row 10.

#27 (VS). Using P10 we first show that the two 4's must be disjoint. Let columns 1-3 be 0^{10}, $1^3\ 0^7$, $0^3\ 1^3\ 0^4$. One of the first six rows must have two 1's under the d_2 columns, so suppose the first row is 010110000... . Then (P10) there are four possibilities for the next two rows:

```
010001100     010001100
010000011     010000010

010001100     010001000
010000000     010000100,
```

and none can be completed to row 10 (P10).

#29 (HVS). Each row must have exactly three 1's in columns 1-9 (if there are four 1's then by switching that row either we get #24 or P4 is violated; the total number of 1's is 27). We now consider all possibilities for columns 1-3, and eliminate them using P10.

#25 (XS). We classify rows by their height, which is at most 3 by P4. If there are (at least) three rows of height 3, they must be (using P8, P10)

```
1110000*00
1101000*11
0000111*01
```

From this it follows that there are at most three rows of height 3. The height distributions of the rows are therefore $3^3\ 2^5\ 1^2$, $3^3\ 2^6\ 0$, $3^2\ 2^7\ 1$, or $3^1\ 2^9$. The first two are easily eliminated using P10, but the other two require much more effort. A large number of 10×8 partial arrays appear, all of which are finally eliminated by P11.

#31 (HVS). Similar to #29.

#26 (D). This reduces to #31 by reversing the two rows mentioned in P9.

#35 (D). This reduces to #31 using P9.

#33 (XS). There are 21 possibilities for the 1's in three columns headed 644, the first and last (after transposing) being

1100...		1100000000
1110...	and	0011100000
1110...		0000011100.

We now use P10 repeatedly (sometimes supplemented by P11 when the number of possible partial configurations grows too large) to show that none may be completed to a 10×9 array.

#30 (XS). There are 16 possibilities for the 1's in three d_4 columns, the first and last (after transposing) being

111...		1110000000
111...	and	0001110000
111...		0000001110.

This case may now be completed in the same way as the previous case.

#34 (XS). Similar to #30.

#32 (XS). Similar to #33.

#28 (D). This reduces to #32 using P9.

#36 (D). The first two columns must be 100... and 011..., and switching 2, 3 produces #35.

#40 (XS). We classify rows by their height, and first eliminate by hand the cases when there is a row of height $\geq$ 3. Then there must be 5 rows of height 2 and 5 rows of height 1. The rows of height 2 may be specified by a graph on 5 nodes having 5 edges, with multiple edges permitted. There are 14 such graphs (e.g. a 5-cycle), and the graph determines columns 1-5 of the array. Figure 11 shows the array corresponding to the 5-cycle.

From P4 the total number of 1's in a row is $\leq$ 5, but it is easy to see that the only possibility for a row with five 1's is $0^5\ 1^5$. Otherwise there are at most four 1's per row. For each of the graphs we use P10 to show that there is no way to fill in columns 6-10.

To illustrate the operation of program P10, Figure 11 shows one possibility for the top three rows of columns 6-10 (found earlier by P10). We now ask P10 to consider all possible ways to add up to five more rows, with the constraint that the total number of 1's in a row must not exceed 4. Table IV shows the beginning of the output of this program. The output specifies the partial row that has been added (in this case the part in columns 6-10). Then either it specifies how this array may be eliminated, or it attempts to add one more row. The program will if necessary add five rows, and keeps track of all partial arrays that survive to the end of the search.

#37 (HVS). Note first that there are at most four 1's per row (or else we get an earlier case). The 8 is disjoint from the 4's (or else we get #28), and the two 4's may meet at most once (or else we get sum > 30). (a) Suppose the two 4's meet once (Fig. 12a). Consider the 1's in a d_2 column. If there is a 1 in the first row, then there are no 1's in rows 2-6 (either $\sum\ >\ 30$ or get #29), and so there are three 1's in rows 7-10. Similarly if there is a 1 in row 2. Then if there is more than a single 1 in the d_2 columns in rows 1 and 2, switching the last four rows produces at least #32, out. Suppose there are no 1's in the d_2 columns in rows 1 and 2. Then the numbers of 1's per row in the d_2 columns must be 0033334444 (in this order), and switching the last four row gives #32 again. Therefore there is a single 1 in the d_2 columns in rows 1 and 2 (in column 4, say) and the numbers of 1's per row are (10)3333(3444), with possible rearrangements inside the parentheses. Figure 2a illustrates this distribution of 1's. In rows 3, 4 there may be at most two 1's in any of column 5-10 (else get #29), and similarly in rows 5, 6. So we may complete rows 3, 4 as shown. But then for all completions of rows 5, 6, by switching one of rows 3/4 and one of 5/6 we get a sum > 30. (b) Suppose the 4's are disjoint (Fig. 12b). As in (a), if there is a 1 in the d_2 columns in row 1 then that column must be $1\ 0^6\ 1^3$, and switching rows 1, 8, 9, 10 yields sum ≥ 34. So the first row is 10^9. In rows 2-4 there is at most one 1 in the d_2 columns (else get #29), and similarly in rows 5-7. So there are most 14 1's in rows 1-7 in the d_2 columns, and at most 12 in rows 8-10. But we need $7 \cdot 4 = 28$ 1's in the d_2 columns, a contradiction.

#38 (D). The 6's are disjoint (else we get #26), and do not meet the 4 (else get #35). So the first three columns are disjoint, say $1^2\ 0^8$, $0^2\ 1^2\ 0^6$, $0^4\ 1^3\ 0^3$. If there are two 1's in a d_2 column in rows 1-4 then we get either #26 or 31 by switching these two rows. Similarly if there are two 1's in a d_2 column in rows 5-7. So there are at most 14 1's in the top 7 rows in the d_2 columns, therefore at least 14 1's in the last 3 rows, hence a row with five 1's. This switches to #25.

#39 (XS). Similar to #33.

This completes the proof of Theorem 6.

The programs were run at Bell Labs on an IBM 3081K and a Cray X-MP. The total computing time needed is probably less than three hours (we actually used much more than this, before discovering efficient methods of tackling the problem).

List of Plate Captions

List of Figure Captions

Figure 1. Extremal sets of lights, for $n = 3, \ldots, 10$.

Figure 2.

Figure 3.

Figure 4. Partial array used by program P10.

Figure 5.

Figure 6.

Figure 7.

Figure 8.

Figure 9.

Figure 10.

Figure 11.

Figure 12.

Figure 2

(a)

8	2	2	2	2	2	0	0
	1	0	0	0	0		
	1	1	0	0	0		
	1	0	1	0	0		
	0	1	0	1	0		
	0	1	0	0	1		
	0	0	1				
	0	0	1				
	0	0	0				

(b)

4	4	4	2	2	2	0	0
1	0	0					
1	0	0					
0	1	0					
0	1	0					
0	0	1					
0	0	1					
0	0	0	1	1	1		
0	0	0	1	1	1		

Figure 3

(a)

7	5	3	3	3	1	1	1	1
		1	1	0	1	0	0	0
		1	0	1	0	1	0	0
		0	1	1	0	0	1	0

(b)

5	5	3	3	3	3	1	1	1
1	0	1	0	0	0	0	1	0
1	0	0	1	0	0	0	0	1
0	1	0	0	1	0			
0	1	0	0	0	1			

(c)

5	5	3	3	3	3	1	1	1
1	0	1	0	0	0	0	1	0
1	0	0	1	0	0	0	0	1
0	1	0	0	1	0		0	0
0	1	0	0	0	1	1	1	1

(d)

3	3	3	3	3	3	3	3	1
1	1	1	0	0	0	0	0	1
0	0	0	(1	1	0	0	0)	1
0	0	0	(1	1	0	0	0)	1
0	0	0	(1	1	0	0	0)	1

Figure 5.

(a)

10	6	4	4	2	2	2
0	1	0	0	1	0	0
0	1	0	0			
0	0	1	1			
0	0	1	1			

(b)

10	6	4	4	2	2	2
0	1	0	0	1	0	0
0	1	0	0			
0	0	1	1			
0	0	1	0			
0	0	1	0			
0	0	0	1			

(c)

10	6	4	4	2	2	2
0	1	0	0	1	0	0
0	1	0	0	0	1	0
0	0	1	0			
0	0	1	0			
0	0	1	0			
0	0	0	1			
0	0	0	1			
0	0	0	1			

Figure 6

(a)

	8	8	4	4
1	1	0	0	0
2	0	1	0	0
3	0	0	1	1
4	0	0	1	1

(b)

	8	8	4	4	2	2	2
1	1	0	0	0	1	0	0
2	0	1	0	0			
3	0	0	1	1	0	0	0
4	0	0	1	0			
5	0	0	1	0			
6	0	0	0	1			
7	0	0	0	1			

(c)

	8	8	4	4	2	2	2
1	1	0	0	0	1	0	0
2	0	1	0	0			
3	0	0	1	0			
4	0	0	1	0			
5	0	0	1	0			
6	0	0	0	1			
7	0	0	0	1			
8	0	0	0	1			

Figure 8

(a)

6	6	6	4	4	2	2
1	0	0	1	0	0	0
1	0	0	0	1	0	0
0	1	0	0	0	1	0
0	1	0	0	0	0	1
0	0	1	0	0	1	0
0	0	1	0	0	0	1
0	0	0	1	0	1	0
0	0	0	1	0	0	1
0	0	0	0	1	1	0
0	0	0	0	1	0	1

(b)

6	6	4	4	4	4	2
1	0	0	0	0	0	1
0	1	0	0	0	0	1
0	0	1	0	0	0	1
0	0	0	1	0	0	1
1	0	0	0	1	0	0
0	1	0	0	0	1	0
0	0	1	0	1	0	0
0	0	1	0	0	1	0
0	0	0	1	1	0	0
0	0	0	1	0	1	0

Figure 9.

(a)

6	4	4	4	4	4	4
1	1	0	0	0	0	0
1	0	1	0	0	0	0
0	1	0	1	0	0	0
0	1	0	0	1	0	0
0	0	1	0	0	1	0
0	0	1	0	0	0	1
0	0	0	1	0	1	0
0	0	0	1	0	0	1
0	0	0	0	1	1	0
0	0	0	0	1	0	1

(b)

8	6	4	4	2	2	2	2
1	0	0	0	0	0	0	0
0	1	0	0	1	0	0	0
0	1	0	0	0	1	0	0
0	0	1	1	0	0	0	0
0	0	1	0	1	0	1	0
0	0	1	0	0	1	1	0
0	0	0	1	1	0	0	1
0	0	0	1	0	1	0	1
0	0	0	0	1	0	1	1
0	0	0	0	0	1	1	1

Figure 10

(a)

8	4	4	4	4	2	2	2
1	1	0	0	0	0	0	0
0	1	1	0	0	1	0	0
0	0	1	1	0	0	1	0
0	1	0	0	0	0	1	1
0	0	1	0	1	0	0	1
0	0	0	1	0	1	0	1
0	0	0	1	0	0	0	0
0	0	0	0	1	1	1	0
0	0	0	0	1	0	0	0
0	0	0	0	0	1	1	1

(b)

8	4	4	4	4	2	2	2
1	0	0	0	0	1	1	0
0	1	1	0	0	0	0	0
0	0	1	1	0	1	0	0
0	0	0	1	1	0	0	0
0	1	0	0	1	0	1	0
0	1	0	0	0	1	0	1
0	0	1	0	0	0	1	1
0	0	0	1	0	0	1	1
0	0	0	0	1	1	0	1
0	0	0	0	0	0	0	0

Figure 11

	4	4	4	4	4	2	2	2	2	2
1	1	1	0	0	0	1	0	0	0	0
2	0	1	1	0	0	0	1	0	0	0
3	0	0	1	1	0	0	0	1	0	0
4	0	0	0	1	1					
5	1	0	0	0	1					
6	1	0	0	0	0					
7	0	1	0	0	0					
8	0	0	1	0	0					
9	0	0	0	1	0					
10	0	0	0	0	1					

Figure 12

(a)

	8	4	4	2	2	2	2	2	2	2
1	1	0	0	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0
3	0	1	0	0	1	1	1	0	0	0
4	0	1	0	0	0	0	0	1	1	1
5	0	0	1	0	1	0	0	1	1	0
6	0	0	1	0	0	1	1	0	0	1
7	0	0	0	1	1	1	1	0	0	0
8	0	0	0	1	1	0	0	1	1	0
9	0	0	0	1	0	1	0	1	0	1
10	0	0	0	0	0	0	1	0	1	1

Figure 12 (continued)

(b)

8	4	4	2	2	2	2	2	2	2
1	0	0	0	0	0	0	0	0	0
0	1	0							
0	1	0							
0	1	0							
0	0	1							
0	0	1							
0	0	1							
0	0	0							
0	0	0							
0	0	0							

List of Table Captions

Table I.

n	N	K	R_n	$t[N, K]$
1	1	1	0	0
2	4	3	1	1
3	9	5	2	2
4	16	7	4	3-4
5	25	9	7	5-6
6	36	11	11	8-10
7	49	13	16	12-15
8	64	15	22	16-21
9	81	17	27	21-27
10	100	19	34	27-34

Table II.

#1.	9 9 1 1 1 1 1 1 1	#4.	7 7 5 1 1 1 1 1 1	#9.	5 5 5 5 1 1 1 1 1
#2.	9 7 3 1 1 1 1 1 1	#6.	7 7 3 3 1 1 1 1 1	#11.	5 5 5 3 3 1 1 1 1
#3.	9 5 5 1 1 1 1 1 1	#7.	7 5 5 3 1 1 1 1 1	#13.	5 5 3 3 3 3 1 1 1
#5.	9 5 3 3 1 1 1 1 1	#10.	7 5 3 3 3 1 1 1 1	#14.	5 3 3 3 3 3 3 1 1
#8.	9 3 3 3 3 1 1 1 1	#12.	7 3 3 3 3 3 1 1 1	#15.	3 3 3 3 3 3 3 3 1

Table III. Distributions of column sums for $n = 10$.

#1.	10 10 2 2 2 2 2 0 0 0	#21.	8 4 4 4 4 2 2 2 0 0
#2.	10 8 4 2 2 2 2 0 0 0	#22.	6 6 6 4 2 2 2 2 0 0
#3.	10 6 6 2 2 2 2 0 0 0	#23.	6 6 4 4 4 2 2 2 0 0
#4.	10 6 4 4 2 2 2 0 0 0	#24.	6 4 4 4 4 4 2 2 0 0
#5.	10 4 4 4 4 2 2 0 0 0	#25.	4 4 4 4 4 4 4 2 0 0
#6.	8 8 6 2 2 2 2 0 0 0	#26.	10 6 2 2 2 2 2 2 2 0
#7.	8 8 4 4 2 2 2 0 0 0	#27.	10 4 4 2 2 2 2 2 2 0
#8.	8 6 6 4 2 2 2 0 0 0	#28.	8 8 2 2 2 2 2 2 2 0
#9.	8 6 4 4 4 2 2 0 0 0	#29.	8 6 4 2 2 2 2 2 2 0
#10.	8 4 4 4 4 4 2 0 0 0	#30.	8 4 4 4 2 2 2 2 2 0
#11.	6 6 6 6 2 2 2 0 0 0	#31.	6 6 6 2 2 2 2 2 2 0
#12.	6 6 6 4 4 2 2 0 0 0	#32.	6 6 4 4 2 2 2 2 2 0
#13.	6 6 4 4 4 4 2 0 0 0	#33.	6 4 4 4 4 2 2 2 2 0
#14.	6 4 4 4 4 4 4 0 0 0	#34.	4 4 4 4 4 4 2 2 2 0
#15.	10 8 2 2 2 2 2 2 0 0	#35.	10 4 2 2 2 2 2 2 2 2
#16.	10 6 4 2 2 2 2 2 0 0	#36.	8 6 2 2 2 2 2 2 2 2
#17.	10 4 4 4 2 2 2 2 0 0	#37.	8 4 4 2 2 2 2 2 2 2
#18.	8 8 4 2 2 2 2 2 0 0	#38.	6 6 4 2 2 2 2 2 2 2
#19.	8 6 6 2 2 2 2 2 0 0	#39.	6 4 4 4 2 2 2 2 2 2
#20.	8 6 4 4 2 2 2 2 0 0	#40.	4 4 4 4 4 2 2 2 2 2

Table III. Illustrates results from program P10.

- Row 4 = 00000 to #23 by 234
- Row 4 = 10000 to #33 by 14
- Row 4 = 01000 to #33 by 24
- Row 4 = 00100 to #20 by 34
- Row 4 = 00010 (not out)
    - Row 5 = 00000 to #23 by 125
    - ...
    - Row 5 = 00010 to #20 by 45
    - Row 5 = 00001 (not out)
        - Row 6 = 00000 to #23 by 126
        - Row 6 = 10000 out by switching 126
        - ...
        - Row 6 = 00111 to #33 by 346
    - Row 5 = 11000 to #20 by 15
    - ...
    - ...
    - Row 5 = 00011 to #20 by 45
- Row 4 = 00001 (not out)
    - Row 5 = 00000 to #23 by 125
    - ...
    - ...

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