Low-Dimensional Lattices V:
Integral Coordinates for Integral Lattices*

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Abstract

We say that an *n*-dimensional (classically) integral lattice Λ is *s*-integrable, for an integer *s*, if it can be described by vectors $s^{-1/2}(x_1, \ldots, x_k)$, with all $x_i \in \mathbf{Z}$, in a Euclidean space of dimension $k \geq n$. Equivalently, Λ is *s*-integrable if and only if any quadratic form $f(x)$ corresponding to Λ can be written as s^{-1} times a sum of *k* squares of linear forms with integral coefficients, or again, if and only if the dual lattice Λ^* contains a eutactic star of scale *s*. This paper gives many techniques for *s*-integrating low-dimensional lattices (such as E_8 and the Leech lattice). A particular result is that any 1-dimensional lattice can be 1-integrated with $k = 4$: this is Lagrange's four-squares theorem. Let $\phi(s)$ be the smallest dimension *n* in which there is an integral lattice that is not *s*-integrable. In 1937 Ko and Mordell showed that $\phi(1) = 6$. We prove that $\phi(2) = 12$, $\phi(3) = 14$, $21 \leq \phi(4) \leq 25$, $16 \leq \phi(5) \leq 22$, $\phi(s) \leq 4s + 2$ (*s* odd), $\phi(s) \leq 2\pi e\, s(1 + o(1))$ (*s* even) and $\phi(s) \geq 2 \log\log s / \log\log\log s\, (1 + o(1))$.

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1. The main theorem.

Let $\phi(s)$ be the least rank for which there exists a classically integral quadratic form $f(x)$ such that $sf(x)$ cannot be written as a sum of squares of linear forms with integral coefficients. The main result of this paper is the following.

Theorem 1.

(i) $\phi(1) = 6$,

(ii) $\phi(2) = 12$,

(iii) $\phi(3) = 14$,

(iv) $21 \le \phi(4) \le 25$,

(v) $16 \le \phi(5) \le 22$,

(vi) $\phi(s) \le 4s + 2$, s odd,

(vii) $\phi(s) \le 2\pi e s\ (1 + o(1))$,

(viii) $\phi(s) \ge 2(\log\log s / \log\log\log s)\ (1 + o(1))$.

We discuss these questions geometrically, in terms of lattices, beginning in the next section with the case $s = 1$.

2. The case s = 1; integrating a lattice.

We say that an n-dimensional lattice Λ can be *integrated* if it can be described by vectors $(x_1, \ldots, x_k)$ with all $x_i \in \mathbf{Z}$ in a Euclidean space of dimension $k \ge n$. Obviously a necessary condition for this is that all inner products in Λ be integers, i.e., that Λ is a (classically) integral lattice, and so from now on ‘‘lattice’’ will mean ‘‘classically integral lattice’’. Is this necessary condition also sufficient? Here we treat this as a problem which is interesting in its own right, but mention that we first met it in the computation of group characters (Conway, 1984). In that paper it was shown that the condition does not suffice when $n = 6$.

Theorem 2. The lattice E_6 cannot be integrated.

Proof. If E_6 can be integrated there is a k-dimensional space $\mathbf{R}^k$ with an orthonormal basis $e_1, \ldots, e_k$, containing a copy of E_6 in which every $v \in E_6$ has integral coordinates. The i-th coordinate of $v \in E_6$ is $v \cdot e_i = v \cdot \bar{e}_i$, where $\bar{e}_i$ is the orthogonal projection of e_i onto the 6-dimensional subspace $\mathbf{R}E_6$ spanned by the lattice. So the vectors $\bar{e}_i$ (of norm ≤ 1) have integral inner products with the vectors of E_6 and are therefore in the dual lattice E_6^* (and are not all zero). This is impossible, since the minimal vectors for E_6^* (i.e. the glue vectors for E_6 — see Part I, Conway & Sloane 1988b) have norm $4/3$.

In terms of quadratic forms the integrability condition for Λ is that the corresponding quadratic form be a sum of squares of linear forms with integral coefficients. In this language Theorem 2 was proved (in a different way) by Mordell (1937a), while Ko (1937, 1939) showed that any form of dimension ≤ 6 is either a sum of squares or the norm form of E_6 plus a sum of squares. So in a strong sense E_6 is the simplest non-integrable lattice; in particular all lattices of dimension $n \leq 5$ are integrable (this will be proved in §3). (In (Conway, 1984) the integrability of 4- and 5-dimensional lattices was left open, while the non-integrability of E_6 was shown in yet another way.)

Our proof of Theorem 2 shows the relevance of the following definition. Take $\mathbf{R}^n$ embedded in $\mathbf{R}^k$ for $k \geq n$. Vectors $\bar{e}_1, \ldots, \bar{e}_k \in \mathbf{R}^n$ (with repetitions allowed) are said to form an n-dimensional *eutactic star* (of *scale* 1) if they are the orthogonal projections onto $\mathbf{R}^n$ of an orthonormal basis $e_1, \ldots, e_k$ for $\mathbf{R}^k$ (cf. Hadwiger, 1940; Coxeter, 1973; Seidel, 1978; Conway & Sloane, 1988c). The argument used to prove Theorem 2 shows:

Theorem 3. An n-dimensional lattice Λ is integrable if and only if its dual lattice Λ^*

contains an n-dimensional eutactic star of scale 1.

3. Embedding lattices in maximal ones.

We now turn to the problem of integrating various families of lattices. Plainly any sublattice of an integrable lattice is integrable. So it is natural to study maximal integral lattices.

Here it is appropriate to consider the problem also over the rings $\mathbf{Q}$, $\mathbf{Q}_p$, $\mathbf{Z}_p$ of rationals, p-adic rationals and p-adic integers, since a lattice is maximal over $\mathbf{Z}$ if and only if it is maximal over $\mathbf{Z}_p$ for all p. The quadratic form corresponding to a symmetric matrix over a ring R will be called an R-form.

The relevant theorem gives the principal connection between the theories of rational and integral quadratic forms. This is a standard result from quadratic form theory; apart from the Type I portion it is equivalent to Theorem 91.2 of O'Meara (1971) (see also Hsia, 1978).

Theorem 4. If p is odd there is a unique $\mathbf{Z}_p$-class of maximal $\mathbf{Z}_p$-forms that are $\mathbf{Q}_p$-equivalent to a given one. There is a unique $\mathbf{Z}_2$-class of maximal Type I $\mathbf{Z}_2$-forms, and also a unique $\mathbf{Z}_2$-class of maximal Type II $\mathbf{Z}_2$-forms, that are $\mathbf{Q}_2$-equivalent to a given one.

Proof. For p odd we essentially proved this in Part I: if the determinant is divisible by p^3 then the lattice can be embedded to index p in a larger one, and the same is true in one of the two cases in which p^2 exactly divides the determinant (cf. Part I, Proposition 4). So for a given determinant there are just four possibilities for the p-adic diagonalization of a maximal $\mathbf{Z}_p$-form, namely

$$\begin{aligned} &\text{diag}\ \{*,\ *,\ *,\ *,\ \ldots roman\}\ , \\ &\text{diag}\ \{pa,\ *,\ *,\ *,\ \ldots roman\}\ , \\ &\text{diag}\ \{pb,\ *,\ *,\ *,\ \ldots roman\}\ , \\ &\text{diag}\ \{pc,\ pd,\ *,\ *,\ \ldots roman\}\ , \end{aligned} \tag{1}$$

where $*$ indicates a p-adic unit, and

$$\left[\frac{a}{p}\right] = +1\ , \quad \left[\frac{b}{p}\right] = -1\ , \quad \left[\frac{-cd}{p}\right] = -1\ .$$

By examining their determinants and p-signatures (see SLG (Conway & Sloane 1988a), Chap. 15), we see that these forms exhaust the four different possibilities for the $\mathbf{Q}_p$-class. For $p = 2$ there is a similar but slightly more complicated proof.

Theorem 5. All the maximal Type I forms in a given rational class are in the same genus, and all the maximal Type II forms (if any) are in a unique other genus.

Proof. Immediate from Theorem 3 and the definition of genus.

Theorem 6. Given an n-dimensional classically integral form f, there is a three-dimensional form g such that $f \oplus g$ belongs to any prescribed rational class of $(n + 3)$-dimensional forms that satisfies the obvious condition that its signature should differ from that of f by at most 3.

Proof. Suppose we wish to change the rational class of the given n-dimensional form f to that of some specified $(n + 3)$-dimensional form h. Consider an odd prime p. We can assume that f has been p-adically diagonalized to one of the forms shown in (1), and h to a (possibly) different one with three more p-adic units. It is now easy to check that there is always a three-dimensional form g_p such that

$$f \oplus g_p \sim_{\mathbf{Q}_p} h\ .$$

For example if

$$f \sim_{\mathbf{Q}_p} \text{diag } \{pa, *, \ldots\} ,$$
$$h \sim_{\mathbf{Q}_p} \text{diag } \{pb, *, *, *, *, \ldots roman\} ,$$

we may take

$$g_p \sim_{\mathbf{Q}_p} \text{diag } \{-pa, pb, *\} .$$

There are similar arguments for $p = 2$ and $p = -1$. Since these p-adic forms g_p automatically satisfy the product formula, they are the localizations of a rational three-dimensional form g.

By combining Theorems 5 and 6 we obtain the following result.

Theorem 7. An arbitrary n-dimensional integral form f can be embedded in a maximal $(n + 3)$-dimensional form of any prescribed genus, with the obvious exceptions that a Type I form cannot be embedded in a Type II form, and that the signature cannot change by more than 3.

This theorem has a number of corollaries.

Corollary 8. Given an n-dimensional integral lattice, there is some k-dimensional odd unimodular lattice containing it, with $k \le n + 3$.

Corollary 9. Given an n-dimensional even integral lattice, there is some k-dimensional even unimodular lattice containing it, where k is the smallest multiple of 8 not less than $n + 3$.

In particular, by Corollary 8 any integral lattice of dimension $n \le 5$ can be embedded in the unique $(n + 3)$-dimensional odd unimodular lattice I_{n+3} and so is integrable. This is Ko's (1937) result. The case $n = 1$ is Lagrange's theorem that any positive integer is the sum of at most four squares (Hardy & Wright, 1980, Theorem 369).

We also see by Corollary 9 that any even integral lattice of dimension $n \leq 5$ can be embedded in E_8.

4. Other types of coordinates; *s*-integrability.

Since integral coordinates are not always possible we consider some alternatives. Theorem 6 shows that an n-dimensional lattice can be embedded in the unimodular Lorentzian lattice $I_{n+2,1}$ (which is unique in its genus). We deduce the following result, which strengthens a theorem of Ko (1940).

Theorem 10. Any n-dimensional integral lattice has a basis $v_1 = (m_{1\,1}, \ldots, m_{1\,k-1}\,;\, m_{1\,k}), \ldots, v_n = (m_{n\,1}, \ldots, m_{n\,k-1}\,;\, m_{n\,k})$ in Lorentzian space $\mathbf{R}^{k-1,1}$, where the m_{ij} are integers and $k \leq n + 3$.

For example E_6 can be described in $\mathbf{R}^{8,1}$ by the generator matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0; & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0; & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0; & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0; & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1; & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0; & 1 \end{bmatrix}, \qquad (2)$$

in which the semicolons separate the time-like coordinate.

Theorem 6 also shows that any n-dimensional lattice can be *rationally* embedded in I_{n+3}. Hence:

Theorem 11. (Mordell, 1932) Any n-dimensional integral lattice has a basis $v_1 = (m_{11}, \ldots, m_{1k}), \ldots, v_n = (m_{n1}, \ldots, m_{nk})$, where the m_{ij} are rational and $k \leq n + 3$.

The case $n = 2$ had been established by Landau (1904). For example E_6 has generator matrix

$$\begin{bmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad (3)$$

(cf. Conway & Sloane, 1988a, p. 126). Moreover $k = n + 3$ is sometimes required, e.g. for the lattice with Gram matrix

$$A = \text{diag } \{7, 1, \ldots, 1\} \ . \qquad (4)$$

However, the main topic of this paper is the integration of scaled versions of given quadratic forms and lattices. The quadratic form $f(x)$ will be called *s-integrable* if $sf(x)$ is integrable, i.e. is the sum of squares of integral linear forms. This is equivalent to the problem of expressing the vectors of the corresponding lattice Λ in the form

$$s^{-\frac{1}{2}} (x_1, \ldots, x_k) \qquad (5)$$

where $x_1, \ldots, x_k$ are integers. For example (3) shows that E_6 is 4-integrable. In other words, the lattice Λ is s-integrable if and only if the scaled lattice ${}^s\Lambda$ is integrable. Here ${}^s\Lambda$ is obtained from Λ by multiplying inner products by s; we may regard ${}^s\Lambda$ as consisting of the vectors $s^{\frac{1}{2}} v$, $v \in \Lambda$. Note that $s\Lambda = {}^{s^2}\Lambda$. Then Λ is s-integrable if and only if it can be embedded in ${}^{1/s}\mathrm{I}_k$, or equivalently if and only if

$$ {}^{s}\mathrm{I}_k \subseteq \Lambda^* \oplus \mathbf{R}^{k-n} \ . \tag{6} $$

In matrix terms we ask whether the Gram matrix A satisfies

$$ sA = MM^{tr} \tag{7} $$

for some $n \times k$ integral matrix M (a generator matrix for the lattice in the desired coordinate system.) When any of those equivalent conditions is satisfied, we say that the form or lattice has been s-integrated in k coordinates.

By rescaling Theorem 3 we obtain:

Theorem 3′. An n-dimensional lattice Λ is s-integrable if and only if Λ^* contains an n-dimensional eutactic star of scale s, that is, vectors $u_1 , \ldots , u_k$ (with repetitions allowed) which are the orthogonal projections from a k-dimensional space containing Λ of k mutually orthogonal vectors of norm s.

For future use we note that a necessary and sufficient condition for $u_1 , \ldots , u_k \in \mathbf{R}^n$ to be an n-dimensional eutactic star is that for each $w \in \mathbf{R}^n$

$$ \sum_{i=1}^{k} (w \cdot u_i) u_i = s\, w \tag{8} $$

or (taking inner products with w)

$$ \sum_{i=1}^{k} (w \cdot u_i)^2 = s\, w \cdot w \ . \tag{9} $$

Furthermore a eutactic star has the property that

$$ \sum_{i=1}^{k} u_i \cdot u_i = s\, n \ . \tag{10} $$

The function $\phi(s)$ of §1 is then the least dimension n of any integral lattice that is not s-integrable. Integrability (the case $s = 1$) was treated in §§2 and 3, and the results of Mordell and Ko show that $\phi(1) = 6$. The following sections deal with larger values of s.

Remarks. Mordell and Ko investigate this problem (for $s = 1$) in terms of expressing a quadratic form as a sum of squares, and call it a ‘‘Waring’s problem for quadratic forms’’. In this language it is related to Hilbert’s seventeenth problem on the representation of functions as sums of squares (Hilbert, 1888; Landau, 1904; Taussky, 1970; Pfister, 1976). Other references are Braun, 1938; Erdös & Ko, 1939; Hunsucker, 1971; Ko, 1936; Mordell, 1930, 1937b; Pall & Taussky, 1957.

Yet another way to coordinate lattices (which we do not pursue here) is to use coordinates which are integers in an algebraic number field. For example E_6 can be described by triples of Eisenstein integers (Sloane, 1979; SLG, p. 126), as shown by the embedding ${}^3E_6 \subseteq A_2^3$ (or $A_2^3 \subseteq E_6^*$).

As to possible applications of these results, we mention first that the recent use of lattices in digital communications, in particular in trellis coded modulation schemes (cf. Forney et al. 1984, Calderbank & Sloane, 1987) requires lattices which can be described by simple coordinates. Second, several recent papers in computer science dealing with lattices and their applications, for example to factoring polynomials (cf. Landau, 1987), only consider n-dimensional lattices that are spanned by n vectors with integral coordinates, i.e. are sublattices of $\mathbf{Z}^n$. The present paper suggests more general ways to coordinatize integral lattices. In a paper somewhat related to the present one, (Cremona & Landau, 1988) consider an n-dimensional lattice $\Lambda \subseteq \mathbf{Z}^n$, and investigate the smallest

t such that $t^{-1}\ \Lambda \subseteq \mathbf{Z}^n$.

5. The case s = 2.

Theorem 12. Any integral lattice Λ of dimension $n \leq 11$ is 2-integrable.

Proof. By Corollary 8, Λ can be embedded in one of the 14-dimensional unimodular lattices, namely (see Part I or SLG, p. 416).

$$\mathrm{I}_{14},\ E_8 \oplus \mathrm{I}_6,\ D_{12}^{+} \oplus \mathrm{I}_2,\ E_7^{2+}\ ,$$

so it will suffice to prove that I_2, E_8, D_{12}^{+} and E_7^{2+} are 2-integrable. We explicitly exhibit eutactic stars of scale 2 in these lattices:

$$\mathrm{I}_2 : \qquad \{11,\ 1\ -1\} \tag{11}$$

$$E_8 : \qquad \{1 \pm 1\ 0^6, 0^2 1 \pm 1\ 0^4, 0^4 1 \pm 1\ 0^2, 0^6 1 \pm 1\} \tag{12}$$

$$D_{12}^{+} : \qquad \{1 \pm 1\ 0^{10}, 0^2 1 \pm 1\ 0^8, \ldots, 0^{10} 1 \pm 1\} \tag{13}$$

If we define E_7 to consist of the vectors of E_8 that are perpendicular to $0^6 1 - 1$, then

$$\{1 \pm 1\ 0^6\ ,\ \ 0^2\ 1 \pm 1\ 0^4\ ,\ \ 0^4\ 1 \pm 1\ 0^2\ ,\ \ 0^6 1^2\} \tag{14}$$

is a eutactic star of scale 2 in E_7, and

$$\{1 \pm 1\ 0^6\ 0^8, \ldots, 0^6 1^2 0^8,\ 0^8 1 \pm 1\ 0^6, \ldots, 0^8 0^6 1^2\} \tag{15}$$

is a star in E_7^{2+}.

Remarks. (i) A more detailed argument (which involves some embeddings in lattices of determinant 2) shows that lattices of dimension $n \leq 11$ can be 2-integrated in at most $n + 3$ coordinates — see the Appendix.

(ii) Once a eutactic star has been found in Λ^*, it is easy to exhibit coordinates of the form (5) for Λ. For if $\{u_1, \ldots, u_k\}$ is a eutactic star of scale s in Λ^*, and

$\{ b_1 , \ldots , b_n \}$ is any basis for Λ, then the vectors

$$v_i = s^{-1/2} (b_i \cdot u_1 , \ldots , b_i \cdot u_k) \tag{16}$$

$(i = 1, \ldots , n)$ form a basis that s-integrates Λ. (By definition the entries $b_i \cdot u_j$ are integers, and from (8) and (9) the v_i's have the same inner products as the b_i's). An example may be found in Fig. 1 below.

The argument of Theorem 12 will not extend to higher dimensions, since the next unimodular lattice is A_{15}^+, obtained by adjoining the glue vector [4] to A_{15}, and is not 2-integrable.

Theorem 13. The unimodular lattice A_{15}^+ is not 2-integrable. More generally, the unimodular lattice $A_{m^2-1}^+ = A_{m^2-1} [m]$ is not $(m-2)$-integrable, if $m \geq 3$.

Proof. Since unimodular lattices are self-dual, we must consider a eutactic star $u_1 , \ldots , u_k$ of scale 2 in A_{15}^+. Because the u_i have norm at most 2, each u_i is in the sublattice A_{15} (for the other vectors of A_{15}^+ have norm at least 3). By (9) we have

$$\sum_{i=1}^{k} (w \cdot u_i)^2 = 2 \, w \cdot w , \tag{17}$$

for any $w \in \mathbf{R} A_{15}^+$. But for the particular vector $w = ({}^{15}/_{16}, (-{}^{1}/_{16})^{15})$, which is the glue vector [1] for A_{15}, each term on the left side of (17) is an integer, while the right side is ${}^{15}/_{8}$. The same argument (again choosing $w = [1]$) shows that $A_{m^2-1}^+$ is not $(m-2)$-integrable.

A similar argument shows that the orthogonal complement in $A_{m^2-1}^+$ of an A_2 sublattice also cannot be $(m-2)$-integrated, if $m \geq 3$.

The proof of Theorem 13 illustrates a general principle. We wish to show that Λ

cannot be s-integrated, where $\Lambda \subseteq \Lambda^*$. Suppose the vectors $u \in \Lambda^*$ with $N(u) \leq s$ generate a sublattice $M \subseteq \Lambda$, so the eutactic star $\mathcal{S}$ satisfies

$$\mathcal{S} \subseteq M \subseteq \Lambda \subseteq \Lambda^* \subseteq M^* \ . \tag{18}$$

If we choose w to be in M^* but not in M, then the left side of Eq. (8) is in M, while with luck the right side may be in $M^* \setminus M$, which is a contradiction.

After Theorem 13 we should search inside $A_{15}^{\dagger}$ for a 12-dimensional lattice that cannot be 2-integrated.

Theorem 14. The orthogonal complement in $A_{15}^{\dagger}$ of any 3-dimensional sublattice with Gram matrix

$$\begin{bmatrix} 3 & 2 & 2 \\ 2 & 3 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

is a 12-dimensional lattice of determinant 7 that is not 2-integrable.

Remarks. There are in fact two such 12-dimensional sublattices L_{12} and L'_{12}, respectively perpendicular to the sublattices $L_3 = \langle a, b, c \rangle$ and $L'_3 = \langle a', b', c' \rangle$, where

$$\begin{aligned} a &= {-\tfrac{1}{4}}^{11}\ {\tfrac{3}{4}}^{2}\ -\tfrac{1}{4}\ \tfrac{3}{4}\ \tfrac{3}{4} \\ b &= {-\tfrac{1}{4}}^{11}\ {\tfrac{3}{4}}^{2}\ \tfrac{3}{4}\ -\tfrac{1}{4}\ \tfrac{3}{4} \\ c &= {-\tfrac{1}{4}}^{11}\ {\tfrac{3}{4}}^{2}\ \tfrac{3}{4}\ \tfrac{3}{4}\ -\tfrac{1}{4}\ , \end{aligned} \tag{19}$$

$$\begin{aligned} a' &= {-\tfrac{1}{4}}^{10}\ {\tfrac{3}{4}}^{3}\ \tfrac{3}{4}\ -\tfrac{1}{4}\ -\tfrac{1}{4} \\ b' &= {-\tfrac{1}{4}}^{10}\ {\tfrac{3}{4}}^{3}\ -\tfrac{1}{4}\ \tfrac{3}{4}\ -\tfrac{1}{4} \\ c' &= {-\tfrac{1}{4}}^{10}\ {\tfrac{3}{4}}^{3}\ -\tfrac{1}{4}\ -\tfrac{1}{4}\ \tfrac{3}{4}\ . \end{aligned} \tag{20}$$

They are the lattices $A_{10}\ A_1\ 154_1\ [3\ 1\ \tfrac{1}{22}]$ and $A_9\ A_2\ 210_1\ [3\ 1\ \tfrac{1}{30}]$ of Part I,

Table 2, and contain 112 and 96 minimal vectors respectively. Both are in the genus $I_{12}(7^+)$. We suspect that there may be a result analogous to Ko's (1939) theorem, asserting that L_{12} and L'_{12} are the only minimal non-2-integrable lattices. The complexity of the proof results from the fact that their duals contain several kinds of vectors of norm at most 2, which must be considered for membership in a possible eutactic star.

Proof. Suppose first that $\Lambda = L_{12}$ can be 2-integrated, and let $S \subseteq \Lambda^*$ be a eutactic star of scale 2. The vectors $A = (5a - 2b - 2c) / 7$, $B = (-2a + 5b - 2c) / 7$, $C = (-2a - 2b + 5c) / 7$ span the dual lattice L_3^*. It is easy to check that minimal representatives for the cosets of L_3 in L_3^* are 0, $\pm A$, $\pm(A + B)$, $\pm(A + B + C)$, of norms 0, $^5/_7$, $^6/_7$, $^3/_7$ respectively. It is important that these numbers are less than 1. The minimal norm in the nonzero cosets of A_{15} in A_{15}^+ is 3, and so the minimal norm in the nonzero cosets of Λ in Λ^* is at least $3 - {}^6/_7 > 2$. We conclude that the only vectors in Λ^* of norm ≤ 2 are obtained by projecting minimal vectors of A_{15} onto the 12-dimensional space $V = \mathbf{R}L_{12}$.

There are therefore six types of vectors of norm ≤ 2 in Λ^*, obtained by projecting the following six types of vectors of A_{15} onto V (compare (19)):

type	A_{15} vector before projection							
α	1	-1	0^9	0	0	0	0	0
β	1	0	0^9	-1	0	0	0	0
γ	1	0	0^9	0	0	-1	0	0
δ	0	0	0^9	1	-1	0	0	0
ε	0	0	0^9	1	0	-1	0	0
ζ	0	0	0^9	0	0	1	-1	0

(21)

By forming their inner products with a, b, c we can calculate the norms of the projections of these vectors onto the 3-dimensional space containing L_3, and hence of the projection onto V:

type	α	β	γ	δ	ε	ζ
norm in 3-space	0	$3/7$	$6/7$	0	$5/7$	2
norm in V	2	$11/7$	$8/7$	2	$9/7$	0

The ζ-type vectors can therefore be ignored. Let n_α, n_β, ... denote the number of vectors of type α, β, ... in $\mathcal{S}$.

Let $w = 1^{11} \ \ 11^2 \ \ -11^3$, a vector of norm 616. Eq. (9) yields

$$100\, n_\beta + 144\, n_\gamma + 484\, n_\varepsilon = 1232 \ , \tag{22}$$

and so

$$n_\gamma \le 8 \ , \tag{23}$$

$$n_\beta + n_\gamma \equiv 0 \ (\text{mod } 11) \ . \tag{24}$$

We shall show that $n_\beta \le 2$, which with (23) and (24) implies that $n_\beta = n_\gamma = 0$, making (22) impossible.

Suppose the $u_i \in \mathcal{S}$ are the orthogonal projections of vectors $w_i = u_i + x_i \in W$, where $x_i \in X \subseteq W$ is in the subspace of W orthogonal to V. If $n_\beta \ge 3$ then from (21) there are two β-type vectors w_j and w_k with a common coordinate 1 and so inner product at least 1. The projections of these vectors have $u_j \cdot u_j = u_k \cdot u_k = {}^{11}/_7$, $u_j \cdot u_k \geq {}^4/_7$, and so $x_j \cdot x_j = x_k \cdot x_k = {}^3/_7$, $x_j \cdot x_k \leq -{}^4/_7$, which is impossible.

The proof that L'_{12} cannot be 2-integrated is similar, although we use a different test vector, $w = 3^{10}\ 5^3\ -15^3$, which produces the equation

$$n_\beta + 81 n_\gamma + 100 n_\varepsilon = 420 , \tag{25}$$

and so $n_\gamma \leq 5$, $n_\varepsilon \leq 4$ and

$$n_\beta + n_\gamma \equiv 0 \pmod{20} .$$

By considering the vectors x_i, as in first part of the proof, we find that $n_\beta \leq 3$. These constraints imply that $n_\beta = n_\gamma = 0$, and now (25) is impossible. This completes the proof of Theorem 14.

Thus we have established $\phi(2) = 12$.

6. The case s = 3.

Theorem 14. Any integral lattice Λ of dimension $n \leq 13$ is 3-integrable.

Proof. After Corollary 8 it suffices to 3-integrate the 16-dimensional odd unimodular lattices, namely

$$\mathrm{I}_{16},\ E_8 \oplus \mathrm{I}_8,\ D_{12}^{+} \oplus \mathrm{I}_4,\ E_7^{2+} \oplus \mathrm{I}_2,\ A_{15}^{+} \oplus \mathrm{I}_1,\ D_8^{2+} .$$

This is accomplished by the following eutactic stars of scale 3 (the notation is explained more fully below):

I_4 : $\{0+++,\ +(0+-)\}$

$E_8 \oplus \mathrm{I}_4$: Fig. 1a

D_{12}^{+} : $\{\pm\frac{1}{2}^{12}\}$ (12 vectors with signs forming a Hadamard matrix H_{12})

$E_7^{2+} \oplus \mathrm{I}_2$: Fig. 2

For $A_{15}^{+} \oplus \mathrm{I}_1$ we use the star

$$\begin{gathered}\{(\tfrac{3}{4}\ \tfrac{3}{4}\ -\tfrac{1}{4}\ \tfrac{3}{4}\ -\tfrac{1}{4}\ -\tfrac{1}{4}\ \cdots\ -\tfrac{1}{4})\ -\tfrac{1}{4}\ -\tfrac{1}{4}\ -\tfrac{1}{4}\ 0\ , \\ 0\ 0\ 0\ 0\ 0\ 0\ \cdots\ 0\ (\ 0\ +\ -\ )\ + \quad (16\ \text{vectors})\ .\end{gathered} \tag{26}$$

(We label the 16 coordinates of A_{15}^{+} by $0,1,\ldots,15$. On coordinates $0,\ldots,12$ we use 13 cyclic shifts of a vector with $\frac{3}{4}$ at positions 0,1,3,9 and $-\frac{1}{4}$ elsewhere, corresponding to the lines of a projective plane of order 3.)

For D_8^{2+} we use the star consisting of the rows of the matrix

$$\begin{bmatrix} \frac{1}{2}\,H_8 & I_8 \\ -I_8 & \frac{1}{2}\,H_8 \end{bmatrix} . \tag{27}$$

Notation. When we specify eutactic stars, all powers of the permutations indicated by the parentheses are to be applied, bars indicate negative numbers, + and − not followed by a positive digit mean +1 and −1, I_n is an identity matrix of order n and H_n is a Hadamard matrix of order n.

Remarks. (i) Fig. 1b displays a generator matrix for E_8 corresponding to the eutactic star in Fig. 1a (see the remarks following Theorem 12).

(ii) The two *even* 16-dimensional unimodular lattices are E_8^2 and D_{16}^{+}. E_8^2 can be 3-integrated (in 24 coordinates). On the other hand we have:

Theorem 15. The unimodular lattice D_{16}^{+} cannot be 3-integrated. More generally, the lattice $D_{4s+4}^+ = D_{4s+4}$ [1] cannot be s-integrated, if s is odd.

Remark. This result includes that fact that $E_8 = D_8^+$ cannot be 1-integrated.

Proof. Apply the ‘‘general principle’’ of §5 (see (18)), taking $w = [1]$.

It is now natural to look for a 14-dimensional non-3-integrable lattice inside D_{16}^+.

Theorem 16. The lattice (denoted by $D_{16}^+ \setminus A_2$) orthogonal to an A_2 sublattice of D_{16}^+ is not 3-integrable. More generally, $\Lambda = D_{4s+4}^+ \setminus A_2$ is a $(4s+2)$-dimensional lattice of determinant 3 that is not s-integrable, if s is odd.

Proof. Let the A_2 lattice be generated by vectors $(+ \; - \; 0)$ on the last three coordinates. Then Λ consists of the vectors $u \in D_{4s+4}^+$ in which the last three coordinates are equal, and Λ^* consists of the projections $\bar{u}$, $u \in D_{4s+4}^+$, projected so as to make the last three coordinates equal. We can write $\bar{u} = u - v$, where v is a minimal representative of a coset of A_2^* / A_2, with $N(v) = 0$ or ${}^2\!/_3$. Hence $|N(u) - N(\bar{u})| \leq {}^2\!/_3$. But if $u \notin D_{4s+4}$ we have $N(u) \geq s + 1$, and so $N(\bar{u}) \geq s + {}^1\!/_3$. So a eutactic star $S \subseteq D_{4s+4}^+$ of scale s must in fact be contained in D_{4s+4}, and again we obtain a contradiction by applying the ‘‘general principle’’ with $w = [1]$.

Remark. This includes the fact that $E_6 = E_8 \setminus A_2$ cannot be 1-integrated, and also gives us a 14-dimensional lattice (the lattice $D_{13}\ 12_1\ [1\ {}^1\!/_4]$ of Part I, Table 2) that cannot be 3-integrated, a 22-dimensional lattice that cannot be 5-integrated, etc.

We have now proved that $\phi(3) = 14$.

7. The case s = 4.

Theorem 17. Any 24-dimensional even unimodular lattice is 4-integrable in 24 coordinates.

Proof. There are 24 such lattices, the Niemeier lattices, one with minimal norm 4 (the Leech lattice Λ_{24}) and 23 with minimal norm 2 (see SLG, Chap. 16). Table 1 gives eutactic stars of scale 4 in these lattices. The notation follows that of Chap. 16 of SLG. In particular, except for A_2^{12}, we use the same glue codes as on p. 407 of SLG.

The symbols describing the eutactic stars in Tables 1 and 2 have the following meaning.

$-$	Mutually orthogonal norm 4 vectors $\{ (2 \ 0^{n-1}) \}$ in a maximal D_n contained in this component. This is a D_n, D_5, D_6, D_8 if the component is a D_n, E_6, E_7, E_8 respectively.
$+$	In an A_n component, mutually orthogonal triples of norm 4 vectors

$$\begin{array}{ccc} +(+--) & 0^4 & \cdots \\ 0^4 & +(+--) & \cdots \\ \cdots & \cdots & \cdots \end{array} . \qquad (28)$$

	In a D_4 component, the vectors $++++$ and $+(+--)$.
$*$	Mutually orthogonal norm 2 vectors in the appropriate component.
A glue word	Mutually orthogonal norm 4 vectors of a type indicated by the glue word.

Each subset of vectors is chosen to be orthogonal to all other vectors in the star. The last column of the table gives the number of vectors in each subset. For example the entries ''+'' and ''5'' for A_{24} describe the eutactic star

$$
\begin{array}{cccccccl}
+(+\ -\ -) & 0000 & 0000 & \cdots & 0000 & 0 & & \\
0000 & +(+\ -\ -) & 0000 & \cdots & 0000 & 0 & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & . & & \\
0000 & 0000 & 0000 & \cdots & +(+\ -\ -) & 0 & (18\ \text{vectors}) & \\
{}^{4}/_{5}{}^{4} & -{}^{1}/_{5}{}^{4} & -{}^{1}/_{5}{}^{4} & \cdots & -{}^{1}/_{5}{}^{4} & {}^{4}/_{5} & & \qquad (29) \\
-{}^{1}/_{5}{}^{4} & {}^{4}/_{5}{}^{4} & -{}^{1}/_{5}{}^{4} & \cdots & -{}^{1}/_{5}{}^{4} & {}^{4}/_{5} & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & . & & \\
-{}^{1}/_{4}{}^{4} & -{}^{1}/_{5}{}^{4} & -{}^{1}/_{5}{}^{4} & & {}^{4}/_{5}{}^{4} & {}^{4}/_{5} & (6\ \text{vectors}) &
\end{array}
$$

It is easy to verify that the specified sets are indeed eutactic stars. This completes the proof of Theorem 17.

Theorem 18.

Any unimodular lattice of dimension $n \le 23$ is 4-integrable in n coordinates.

Proof. Although the lattice D_n is often defined only for $n \ge 4$, it is convenient here (as in Part I) to let $D_1 \cong 4_1$ (generated by a vector of norm 4), $D_2 \cong A_1 \oplus A_1$, $D_3 \cong A_3$. Then (see SLG, Chap. 16) any n-dimensional unimodular lattice Λ_n (for $n \le 23$) can be obtained by taking a suitable sublattice D_m (where $m = 24 - n$) in a Niemeier lattice L_{24}, and proceeding as follows. If V is the m-dimensional space containing D_m, we can write any $v \in L_{24}$ as $v = v' + v''$, where $v' \in V$ and v'' is perpendicular to V. Then

$$\Lambda_n = \{ v'' : v \in L_{24},\ v' = [0] \text{ or } [2] roman \} ,$$

where [0], [1], [2], [3] are coset representatives for D_m in D_m^* (cf. Appendix to Part 1). All such Λ_n are described in Table 2, which is based on Table 16.7 of SLG. The first column gives the parent Niemeier lattice L_{24}, and the remaining columns specify m mutually orthogonal norm 4 vectors in a D_m, where $m = 24 - n$, in the notation described after Theorem 17.

We construct a eutactic star of scale 4 for each lattice Λ_n in Table 2, by modifying the star $\mathcal{S} = \{u_1, \ldots, u_{24}\}$ given in Table 1 for the parent Niemeier lattice. For entries marked 'a' in Table 2 this is easy. The D_m used to construct Λ_n is generated by vectors belonging to $\mathcal{S}$, and we obtain the star for Λ_n simply by omitting these vectors from $\mathcal{S}$.

For entries marked 'b' the D_m contains vectors of the shape ++++, $+(+--)$ which are not in $\mathcal{S}$. In these cases we modify $\mathcal{S}$ by

$$\begin{array}{l} \text{replacing the vectors} \\ u_1 = 2000,\ u_2 = 0200,\ u_3 = 0020,\ u_4 = 0002 \\ \text{by either} \\ v_1 = ++++,\ v_2 = ++--,\ v_3 = +-+-,\ v_4 = +--+ \\ \text{or} \\ v_1' = -+++,\ v_2' = -+--,\ v_3' = --+-,\ v_4' = ---+ \,. \end{array} \tag{30}$$

This is permissible as long as one of the v_i's, say $v_1 = \tfrac{1}{2}(u_1 + u_2 + u_3 + u_4)$, is in D_m (and therefore in L_{24}). For then $v_2 = v_1 - u_3 - u_4$, v_3, v_4 are also in L_{24} and are orthogonal to v_1 and each other.

The modified $\mathcal{S}$ now contains generators for D_m, and we may proceed as before.

Eutactic stars for the remaining (unmarked) entries in Table 2 are found by certain modifications of this argument. (These are all 23-dimensional lattices, defined by a norm 4 vector $v_4 \in L_{24}$.)

Most of the cases can be dealt with by applying the following principle.

> If the inner products of $v_4 \in L_{24}$ with the vectors of the eutactic star for L_{24} are either (*i*) $\pm 2^4\ 0^{20}$ or (*ii*) $\pm 2^3\ \pm 1^4\ 0^{17}$ then Λ_{23} can be 4-integrated. (31)

In case (i) the hypothesis implies that for some choice of signs $v_4 = {}^1/_2 (\pm u_i \pm u_j \pm u_k \pm u_l)$ is in L_{24}, and so we may apply the transformation (30) to modify the star so as to include v_4. By omitting v_4 we then obtain the star for Λ_{23}. In case (ii) we have $v_4 = {}^1/_2 (\pm u_i \pm u_j \pm u_k) + {}^1/_4 (\pm u_p \pm u_q \pm u_r \pm u_s) \in L_{24}$, hence ${}^1/_2 (\pm u_p \pm u_q \pm u_r \pm u_s) \in L_{24}$, and two successive applications of (30) puts v_4 into the star.

Consider for example the case when $L_{24} = D_{16} E_8$ and v_4 is described by the symbol 10. The eutactic star in L_{24} and the vector v_4 are:

$$\begin{array}{ccccccc} & (2000) & 0000 & 0000 & 0000 & 0000 & 0000 \\ * & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & 0000 & 0000 & 0000 & 0000 & 0000 & (2000) \\ \\ v_4 & {}^1/_2{}^4 & {}^1/_2{}^4 & {}^1/_2{}^4 & {}^1/_2{}^4 & 0^4 & 0^4 \end{array} \quad (32)$$

Unfortunately the inner products of v_4 with the star are $\pm 1^{16}\ 0^8$. But if we first modify the star by applying (30) to each of the first four blocks of four coordinates, the inner products become $\pm 2^4\ 0^{20}$, and we can apply (31i).

The case $L_{24} = A_{17} E_7$, $v_4 = 31$ illustrates the application of (31ii), although we must first use (30) to modify four vectors (2000) belonging to the E_7 part of the star.

The remaining cases can be handled by the following principle.

> Suppose the inner products of $v_4 \in L_{24}$ with the vectors of the star for L_{24} are $\pm 1^8 \pm 2^2\ 0^{14}$, and suppose there is another vector $w_4 \in L_{24}$ whose inner products with the star are $\pm 2^4\ 0^{20}$. Then if the tetrad of vectors whose inner products with w_4 are ± 2 is a subset of

the octad of vectors whose inner products

with v_4 are $\pm 1^8$, then Λ_{23} can be 4-integrated. (33)

(For after a modification of type (30) on this tetrad, the inner products of v_4 with the star vectors will be $\pm 2^3 \ \pm 1^4 \ 0^{17}$.)

We apply (33), for example, when $L_{24} = A_{11} D_7 E_6$, $v_4 = 330$, taking $w_4 = 222$.

This completes the proof of Theorem 18, and shows that $\phi(4) \geq 21$. In the other direction we shall first prove:

Theorem 19. The 27-dimensional unimodular lattice $\Lambda = A_{26} \ 3_1 \ [3 \ {}^1\!/_3]$ is not 4-integrable.

Proof. The following are all the nonzero vectors in Λ of norm ≤ 4:

type	vector	norm
α	$+ \ - \ 0^{25} \ 0$	2
β	$+ \ + \ - \ - \ 0^{23} \ 0$	4
γ	$0^{27} \ {}^3\!/_\lambda$	3
δ	${{}^8\!/_9}^3 \ {-{}^1\!/_9}^{24} \ {}^1\!/_\lambda$	3
ε	${{}^8\!/_9}^3 \ {-{}^1\!/_9}^{24} \ -{}^2\!/_\lambda$	4

where $\lambda = \sqrt{3}$. By taking $w = 0^{27} \ \sqrt{3}$ in Eq. (9) we obtain

$$9 n_\gamma + n_\delta + 4 n_\varepsilon = 12 \ ,$$

which implies that $(n_\gamma , \ n_\delta , \ n_\varepsilon)$ is one of (0,0,3), (0,4,2), (0,8,1), (0,12,0) or (1,3,0).

We now use the same w in Eq. (8). Consider the first 27 coordinates of the resulting identity. For a δ-type vector u, the contribution from $(w \cdot u) u$ to the sum in (8) is $-{}^1\!/_9$ (modulo 1), the contribution from an ε-type vector is ${}^2\!/_9$ (modulo 1), and the α-, β- and γ-type vectors contribute 0. The total must be $4w$, which vanishes in the first 27 coordinates. Thus

$$\sum_{(n_\delta)} -{}^1\!/_9 + \sum_{(n_\varepsilon)} {}^2\!/_9 \equiv 0 \pmod 1 ,$$

which implies that $(n_\gamma , n_\delta , n_\varepsilon) = (0,4,2)$.

Call any of the initial 27 coordinates ‘‘special’’ if a δ- or ε-type vector takes the value ${}^8\!/_9$ there. There are at most $(4 + 2)3 = 18$ special coordinates. Take $w = -{}^1\!/_{27}{}^{26}\ {}^{26}\!/_{27}{}^{1}\ 0$ in (8), where the ${}^{26}\!/_{27}$ is not in a special coordinate, and consider the value of that coordinate in the resulting identity. The left side is ${}^1\!/_{81}$ (mod 1), while on the right we obtain $4 \cdot {}^{26}\!/_{27} = {}^{104}\!/_{27}$. This is a contradiction.

Hence $\phi(4) \leq 27$. We can improve this slightly.

Theorem 20.

The lattice $A_{26}\ 3_1\ [3\ {}^1\!/_3] \setminus A_2$ is not 4-integrable. This is the 25-dimensional determinant 3 lattice

$$A_{23}\ (3\ {}^1 3)_2\ [3\ {}^1\!/_8\ {}^7\!/_8]\ .$$

The proof is similar to that of Theorem 19 and is omitted.

Combining Theorems 18 and 20 we see that $21 \leq \phi(4) \leq 25$.

8. The case s = 5.

Theorem 21. Any integral lattice Λ of dimension $n \leq 15$ is 5-integrable.

Proof. By Corollary 8, Λ can be embedded in one of the 18-dimensional unimodular lattices, which are (from SLG p. 416).

$$\mathrm{I}_{18}\,,\; E_8 \oplus \mathrm{I}_{10}\,,\; D_{12}^{+} \oplus \mathrm{I}_6\,,\; E_7^{2+} \oplus \mathrm{I}_4\,,\; A_{15}^{+} \oplus \mathrm{I}_3\,,$$
$$D_{16} \oplus \mathrm{I}_2\,,\; E_8^2 \oplus \mathrm{I}_2\,, D_8^2 \oplus \mathrm{I}_2\,,\; A_{11} E_6 \oplus \mathrm{I}_1\,,$$
$$(A_{17} A_1)^{+}\,,\; (D_{10} E_7 A_1)^{+}\,,\; D_6^{3+}\,,\; A_9^{2+}\;.$$

The following eutactic stars of scale 5 then establish the theorem.

$$\mathrm{I}_2 : \qquad \{12,\ \bar{2}1\} \tag{34}$$

$$E_8 : \qquad \{\ (\ C\ 0^5\ )\ \}\quad (10\ \text{vectors}) \tag{35}$$

where each C is to be replaced by the five vectors

$$0 + - - +, \quad + 0 + - -, \quad - + 0 + -, \quad - - + 0 +, \quad + - - + 0 \tag{36}$$

$$D_{12}^{+} : \{ {}^{3}\!/_{2}\ {}^{1}\!/_{2}{}^{11},\ {}^{1}\!/_{2}\ ({}^{3}\!/_{2}\ {}^{1}\!/_{2}\ -{}^{1}\!/_{2}\ {}^{1}\!/_{2}\ {}^{1}\!/_{2}\ {}^{1}\!/_{2}\ -{}^{1}\!/_{2}\ -{}^{1}\!/_{2}\ -{}^{1}\!/_{2}\ {}^{1}\!/_{2}\ -{}^{1}\!/_{2}) \} \tag{37}$$

E_7^{2+} :

$$\{C0^3 0^8\,,\ 0^8 C0^3\,,\ {}^{1}\!/_{4}{}^{5}({}^{1}\!/_{4}\ -{}^{3}\!/_{4}\ -{}^{3}\!/_{4})\,{}^{1}\!/_{4}{}^{5}({}^{1}\!/_{4}\ -{}^{3}\!/_{4}\ -{}^{3}\!/_{4})\,,$$
$${}^{1}\!/_{4}{}^{5}({}^{1}\!/_{4}\ -{}^{3}\!/_{4}\ -{}^{3}\!/_{4})\ -{}^{1}\!/_{4}{}^{5}(-{}^{1}\!/_{4}\ {}^{3}\!/_{4}\ {}^{3}\!/_{4})\,,\ 0^5(+\ -\ 0)\ 0^8\,,\ 0^8 0^5(+\ -\ 0)\ \}\ (22\ \text{vectors}) \tag{38}$$

$A_{15}^{+} \oplus \mathrm{I}_3$:

$$\begin{gathered}\{C0^{10}0\ +\ 00,\ 0^5 C0^5 0\ 0\ +\ 0,\ 0^{10} C0\ 00\ +\ ,\\ ({}^{3}\!/_{4}{}^{5}\ -{}^{1}\!/_{4}{}^{5}\ -{}^{1}\!/_{4}{}^{5})\ -{}^{5}\!/_{4}\ 000\}\ (18\ \text{vectors})\end{gathered} \tag{39}$$

D_8^{2+} :

$$\begin{gathered}\{C0^3 0^8\,,\ 0^8 C0^3\,,\ {}^{1}\!/_{2}{}^{8} 0^5 - 1^3\,,\\ {}^{1}\!/_{2}{}^{5}({}^{1}\!/_{2}\ -{}^{1}\!/_{2}\ -{}^{1}\!/_{2})\ 0^5\ (-\ +\ +\ +\ ),\ 0^5 - 1^3\ {}^{1}\!/_{2}{}^{5} - {}^{1}\!/_{2}{}^{3}\,,\\ 0^5(-\ +\ +\ +)\ {}^{1}\!/_{2}{}^{5}(-{}^{1}\!/_{2}\ {}^{1}\!/_{2}\ {}^{1}\!/_{2})\}\quad (18\ \text{vectors})\end{gathered} \tag{40}$$

D_{16}^{+} :

$$\begin{gathered}\{(C0^5 0^5)0,\ {}^{1}\!/_{2}{}^{15}\ {}^{1}\!/_{2}\,,\\ ({}^{1}\!/_{2}{}^{5}\ -{}^{1}\!/_{2}{}^{5}\ -{}^{1}\!/_{2}{}^{15})\,{}^{1}\!/_{2}\,,\ 0^{15}\ 2\}\ (20\ \text{vectors})\end{gathered} \tag{41}$$

$(A_{11}E_6)^+ \oplus \mathrm{I}_1$:

$$\{(C0^5)0^2 0^8\ 0,\ 0^{12}\ C0^3\ 1\ ,$$
$$(\tfrac{1}{2}^5\ -\tfrac{1}{2}^5)\tfrac{1}{2}\ -\tfrac{1}{2}\ \tfrac{1}{2}^5\ \tfrac{1}{2}^3\ 0\ ,\ -\tfrac{1}{6}^{10}\ \tfrac{5}{6}^2\ \tfrac{1}{2}^5\ -\tfrac{5}{6}^3\ 0\ ,$$
$$-\tfrac{1}{6}^{10}\ -\tfrac{1}{6}\ \tfrac{11}{6}\ 0^5\ \tfrac{2}{3}^3\ 0\ ,\ -\tfrac{1}{6}^{10}\ \tfrac{11}{6}\ -\tfrac{1}{6}\ -\tfrac{1}{2}^5\ \tfrac{1}{6}^3\ 0\}\ \ (20\ \text{vectors}) \tag{42}$$

$(A_{17}A_1)^+$:

$$\{(C0^5 0^5)0^3 0^2\ ,\ (\tfrac{2}{3}^5\ -\tfrac{1}{3}^5\ -\tfrac{1}{3}^5)\ -\tfrac{1}{3}\ -\tfrac{1}{3}\ \tfrac{2}{3}\ 0^2\ ,$$
$$-\tfrac{1}{6}^{15}\ (-\tfrac{11}{6}\ -\tfrac{1}{6})\ \tfrac{5}{6}(\tfrac{1}{2}\ -\tfrac{1}{2})\ ,\ 0^{15}(10)\overline{1}(\overline{1}1)\ \}\quad (22\ \text{vectors}) \tag{43}$$

$(D_{10}E_7A_1)^+$:

$$\{(C0^5)0^8 0^2\ ,\ 0^{10}C0^3 0^2\ ,\ \tfrac{1}{2}^{10}\ 0^8\ \tfrac{1}{2}\ -\tfrac{1}{2}\ ,$$
$$\tfrac{1}{2}^{10}0^8 -\tfrac{1}{2}\ \tfrac{1}{2}\ ,\ (\tfrac{1}{2}^5 -\tfrac{1}{2}^5)\tfrac{1}{2}^5 -\tfrac{1}{2}\ 0^2 0^2\ ,\ 0^{10}\ \tfrac{1}{2}^5\ \tfrac{1}{2}\ \pm\tfrac{1}{2}\ \pm\tfrac{1}{2}0^2\ , \tag{44}$$
$$0^{10}0^5 2 0^2 0^2\ ,\ 0^{10}0^6 +{}^2 0^2\ (\text{twice})\ ,\ 0^{10}0^8 + -\ (\text{twice})\ \}\ (26\ \text{vectors})$$

D_6^{3+} :

$$\{(C0\ 0^6 0^6)\ ,\ 0^5 2 0^{12}\ ,\ \tfrac{1}{2}^6\ 0^5\ \pm 1\ \pm\tfrac{1}{2}^5\ \mp\ \tfrac{1}{2}\ ,$$
$$-\tfrac{1}{2}^5\ \tfrac{1}{2}\ \pm\tfrac{1}{2}^6\ 0^5\ \pm 1\ ,\ 0^6\ \tfrac{1}{2}^5\ -\tfrac{3}{2}\ \tfrac{1}{2}^5\ -\tfrac{1}{2}\ ,\ 0^6\ \tfrac{1}{2}^6\ -\tfrac{1}{2}^5\ -\tfrac{3}{2}\ \}\quad (22\ \text{vectors}) \tag{45}$$

A_9^{2+} :

$$\{(C0^5 0^5 0^5)\ ,\ \tfrac{1}{2}^5\ -\tfrac{1}{2}^5\ \pm\tfrac{1}{2}^5\ -\tfrac{1}{2}^5\ \}\ (22\ \text{vectors})\ . \tag{46}$$

This completes the proof of Theorem 21, and shows that $\phi(5) \geq 16$.

In the other direction we saw from Theorem 16 that there is a 22-dimensional lattice that cannot be 5-integrated, and so $16 \leq \phi(5) \leq 22$.

9. Larger values of s.

From Theorem 16 we have $\phi(s) \leq 4s + 2$ if s is odd, and from the remark following Theorem 13, $\phi(s) \leq s^2 + 4s + 1$ for all s. For large s the latter can be strengthened.

For s must at least equal the minimal norm of Λ^*. However, n-dimensional unimodular lattices exist with minimal norm μ satisfying

$$\mu \geq \frac{n}{2\pi e} (1 + o(1))$$

(Milnor & Husemoller, 1973, pp. 46-47; SLG, Chap. 7, Theorem 25). Hence

$$\phi(s) \leq 2\pi e\, s(1 + o(1)) \ .$$

Before giving a lower bound on $\phi(s)$ we first state two lemmas.

Lemma 22. If a lattice Λ is s-integrable in k coordinates and t-integrable in l coordinates, then it is $(s+t)$-integrable in $k + l$ coordinates.

Proof. The hypothesis tells us that ${}^s\Lambda$ and ${}^t\Lambda$ can be given integral coordinates. By juxtaposition we obtain $k + l$ integral coordinates for ${}^{s+t}\Lambda$.

Corollary 23. (i) E_8 is s-integrable for all $s \geq 2$. (ii) The Leech lattice Λ_{24} is s-integrable for all $s \geq 4$.

Remark. The number of coordinates used is reduced in the Appendix.

Proof. (i) is proved by repeated application of Lemma 22 and the fact that E_8 is 2- and 3-integrable. (ii) follows similarly, using the fact that Λ_{24} is 4-, 5- 6- and 7-integrable. We 4-integrated it in Table 1, and to 6-integrate it we use the star $8^{-1/2} \{ \pm 2^{12} 0^{12}, \ 0^{12} \pm 2^{12} \}$, where the signs form two Hadamard matrices of order 12. To 5- and 7-integrate Λ_{24} we construct it via the ''holy'' constructions of type A_4^6 and A_6^4 respectively (see Conway & Sloane, 1982; SLG, Chap. 24), which produce Λ_{24} as the lattice spanned by the differences of certain vectors f_i and g_w. For a type A_m^n construction there are vectors $f_i = (+ \ - \ 0^{n-1})$ spanning each copy of A_m. The eutactic star $\{ ((0 + - - +) 0^5 0^5 0^5 0^5 0^5) \}$ (30 vectors) is therefore in the A_4^6 version of Λ_{24}, and shows that Λ_{24} can be 5-integrated. Similarly the star $\{ ((0 + + - + - -) 0^7 0^7 0^7) \}$ (28 vectors) is in the A_6^4 version, and shows that Λ_{24} can be 7-integrated.

Of course if a lattice can be s-integrated in k coordinates then it can be $m^2 s$-integrated in k coordinates for any positive integer m. This can sometimes be strengthened:

Lemma 24. If Λ can be s-integrated in $4k$ coordinates then it can be ms-integrated in $4k$ coordinates for any positive integer m.

Proof. In an obvious manner we can represent the $4k$ integral coordinates for ${}^s\Lambda$ by k quaternionic coordinates $(q_1 , \ldots , q_k)$. We replace these by $(qq_1 , \ldots , qq_k)$, where $q = a + bi + cj + dk$ and $a^2 + b^2 + c^2 + d^2 = m$.

To obtain a lower bound on $\phi(s)$ we first prove:

Theorem 25. For any $\varepsilon > 0$ there is an n_0 such that for $n \geq n_0$ any n-dimensional unimodular lattice Λ can be s-integrated for some s satisfying

$$\log s \leq \left[\frac{n}{2} \log n \right] (1 + \varepsilon) \ . \tag{47}$$

Proof. We apply the theorem of Kabatiansky and Levenshtein (1978) (see also SLG, Chap. 9 and p. 20, Eq. (48)), which implies that there is a constant c such that any n-dimensional lattice of determinant d contains a vector of norm μ satisfying

$$\frac{\mu}{n} < c d^{1/n} \ . \tag{48}$$

The first step is to show that Λ contains a mutually orthogonal set of vectors $v_1 , \ldots , v_n$ with norms $\mu_1 , \ldots , \mu_n$ satisfying

$$\log\,(\,\mu_1\,\mu_2\,\cdots\,\mu_n)\;<\;\frac{n^2}{2}\;\log\,n\;(1\,+\,o(1))\;. \tag{49}$$

We set $\Lambda_0 = \Lambda$, an n-dimensional lattice of determinant $d_0 = 1$. In Λ_i ($i = 0, \ldots, n - 1$), an $(n-i)$-dimensional lattice of determinant d_i, by (48) there is a vector v_{i+1} of norm μ_{i+1} satisfying

$$\mu_{i+1}\;<\;c(n\;-\;i)\;d_i^{1/(n\;-\;i)}\;. \tag{50}$$

We set

$$\Lambda_{i+1}\;=\;\{u \in \Lambda_i\;:\;u\,\cdot\,v_{i+1}\;=\;0\}\;,$$

a lattice of determinant

$$d_{i+1}\;=\;\mu_{i+1}d_i\;=\;\mu_1\mu_2\cdots\mu_{i+1}\;. \tag{51}$$

Then we find that

$$d_{i+1}\;<\;c(n\;-\;i)\;d_i^{(n-i+1)/(n-i)}\;,$$

and therefore

$$\begin{aligned}
&d_1\;<\;cn\;,\\
&d_2\;<\;c^{a_2}\;(n-1)^{\frac{n-1}{n-1}}\;n^{\frac{n}{n-1}}\;,\\
&d_3\;<\;c^{a_3}\;(n-2)^{\frac{n-2}{n-2}}(n-1)^{\frac{n-1}{n-2}}\,n^{\frac{n}{n-2}}\;,\\
&\cdots\\
&d_n\;<\;c^{a_n}\;1^1\;2^2\;3^3\;\cdots\;n^n\;,
\end{aligned}$$

where the exponent (a_i) of c in the bound for d_i satisfies

$$a_1\;=\;1,\qquad a_{i+1}\;=\;1\;+\;\frac{n-i+1}{n-i}\;a_i\;.$$

Hence

$$a_{i+1} = \frac{i+1}{n-i} \left[n - \frac{i}{2} \right] , \qquad a_n = \frac{n^2 + n}{2} .$$

Thus

$$\log d_n < \frac{n^2 + n}{2} \log c + \sum_{r=1}^{n} r \log r ,$$

and so

$$\log d_n < \frac{n^2}{2} \log n \, (1 + o(1)) ,$$

which is (49).

Second, we set $s = d_n = \mu_1 \cdots \mu_n$, and form a eutactic star of scale s in Λ by taking s / μ_i copies of v_i ($i = 1, \ldots, n$). Then (47) follows from (49), completing the proof of Theorem 25.

We can use Theorem 25 and Lemma 22 to show that there is a single s^* such that every n-dimensional odd unimodular lattice can be s^*-integrated. s^* does not exceed the least common multiple of the values of s for the individual lattices, so using $lcm\{1, 2, \ldots, k\} = e^{k(1 + o(1))}$ we obtain

$$\log \log s^* \leq \left[\frac{n}{2} \log n \right] (1 + o(1)) .$$

This implies

$$\phi(s) \geq \frac{2 \log \log s}{\log \log \log s} (1 + o(1)) ,$$

and completes the proof of Theorem 1.

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Appendix. Further examples; reducing the number of coordinates

In this Appendix we give some additional examples of s-integrations and in some cases attempt to reduce the number of coordinates needed.

If a lattice can be 1-integrated then by Lemma 22 it can be s-integrated for all s. This is true of the lattices I_n, A_n and D_n $(n \geq 1)$, whose definitions use integral vectors, and of any integral lattice of dimension $n \leq 5$ (by §3).

The lattice I_2 can be s-integrated in two coordinates whenever s is the sum of two squares: if $s = a^2 + b^2$ we use the star $\{ab, \bar{b}a\}$. Similarly I_4 can be s-integrated for all s in four coordinates: write $s = a^2 + b^2 + c^2 + d^2$, and use the star $\{abcd, b\bar{a}d\bar{c}, c\bar{d}\bar{a}b, dc\bar{b}\bar{a}\}$. Thus

$$ {}^{s}\mathrm{I}_4 \subseteq \mathrm{I}_4 \subseteq {}^{1/s}\mathrm{I}_4 \quad (s = 1,2,...) \ . \tag{55} $$

Since $D_4 \subseteq \mathrm{I}_4$, we deduce that D_{4m} can be s-integrated for all s in $4m$ coordinates $(m \geq 1)$. Similarly the unimodular lattice D_{4m}^{+} can be s-integrated for all even s in $4m$ coordinates (for if s is even the eutactic star for I_4 actually belongs to D_4).

Furthermore, provided a skew Hadamard matrix H_{4m} of order $4m$ exists, D_{4m}^{+} can be $(m+2)$-integrated in $4m$ coordinates. H_{4m} certainly exists for $1 \leq m \leq 18$ (Hall, 1967; Wallis et al. 1972). Let $H_{4m} = I_{4m} + S_{4m}$, where I_{4m} is an identity matrix and S_{4m} is skew-symmetric. Then the rows of

$$ \tfrac{1}{2}(3\, I_{4m} + S_{4m}) \tag{56} $$

form the desired star.

In §9 we observed that since E_8 can be 2-integrated in 8 coordinates (Eq. (18)) and 3-integrated in 12 coordinates (Fig. 1), it follows from Lemma 22 that E_8 can be s-integrated for all $s \geq 2$, although the number of coordinates increases with s. In fact E_8 can be s-integrated for all $s \geq 2$ in at most 17 coordinates. For from Lemma 24 E_8 can be s-integrated for all even s in 8 coordinates. Eq. (35) shows E_8 can be 5-integrated in 10 coordinates. The eutactic stars

$$E_8 : \{ (0 + + - + - -)0 \ , \ {}^1\!/_2{}^8 \ (3 \text{ times}) \ , \ {}^1\!/_2{}^7 \ {-}^3\!/_2 \ , \ 0^7 2 \ \} \ , \qquad (57)$$

$$E_8 : \{ ({}^8\!/_3 \ {-}^1\!/_3{}^8) \ \} \qquad (58)$$

show that E_8 can be 7- and 9-integrated in 12 and 9 coordinates respectively. (In (58) we use the form $A_8[3]$ for E_8.) Finally for odd $s \geq 11$ we write $s = 9 + 2t$ and use Lemma 22 to s-integrate E_8 in at most 17 coordinates.

Incidentally, since E_8 has a endomorphism that multiplies norms by m (for any positive integer m, cf. SLG, Chap. 8), if E_8 can be s-integrated in k coordinates then it can be ms-integrated in k coordinates.

A similar remark applies to the Leech lattice, and combined with the eutactic stars given in §9, shows that Λ_{24} can be s-integrated for all $s \geq 4$ in at most 54 coordinates. We suspect that the numbers 17 and 54 can be considerably improved.

In the other direction we observe from Eq. (10) that if s is odd then at least $sn / (s - 1)$ coordinates are needed to s-integrate an n-dimensional even unimodular lattice. Hence E_8 cannot be 3-integrated in fewer than 12 coordinates, or 5-integrated in fewer than 10 coordinates, etc.

E_6 and E_7 can be s-integrated as sublattices of E_8. In particular E_6 and E_7 can be 2-integrated in 7 coordinates.

In Theorem 11 we showed that any lattice of dimension $n \le 11$ can be 2-integrated. This can be slightly strengthened, to show that (i) any unimodular lattice of dimension $n \le 14$ and any determinant 2 lattice of odd dimension $n \le 13$ can be 2-integrated using n coordinates, and (ii) any integral lattice of dimension $n \le 11$ can be 2-integrated using at most $n + 3$ coordinates.

To prove (i) we exhibit eutactic stars of scale 2 in the duals of the indecomposable odd-dimensional lattices of determinant 2 (for a list of these lattices see Part I or SLG, Tables 16.7, 15.8).

lattice	eutactic star
A_1	$\{\sqrt{2}\}$
E_7	Eq. (14)
$D_{10}\ A_1[11]$	$v_2^{\perp}$ in the eutactic star for D_{12}^{+} (see (13))
$D_6 E_7[11]$	$v_2^{\perp}$ in the eutactic star for $E_7^2[11]$ (see (15))

(The star for $D_{10}\ A_1[11]$ for example is obtained from that for D_{12}^{+}, using the fact that $D_{10}\ A_1\ [11]$ is the subset of D_{12}^{+} orthogonal to a fixed vector of norm 2.)

To prove (ii), let Λ be an arbitrary n-dimensional integral lattice. In view of Eq. (6) we wish to show that ${}^2\mathrm{I}_k \subseteq \Lambda^* \oplus \mathbf{R}^3$, $k = n + 3$. A maximal odd integral lattice containing ${}^2\mathrm{I}_k$ is unimodular if k is even, or belongs to the odd genus $\mathrm{I}_k(2)$ of determinant 2 lattices if k is odd (compare Theorem 5). Therefore, using Theorem 7, we embed Λ in an odd lattice of dimension $n + 3$ and determinant 1 (if n is odd) or determinant 2 (if n is even), and deduce (ii) from (i).

Similarly we could slightly strengthen Theorem 14, to say that any odd unimodular lattice of dimension $n = 4m \leq 16$ can be 3-integrated using n coordinates, and any integral lattice of dimension $n \leq 13$ can be 3-integrated using at most $n + 6$ coordinates. But $n + 6$ seems quite weak.

List of Figure Captions

Figure 1 (a) Eutactic star of scale 3 in $E_8 \oplus \mathrm{I}_4$. (b) Corresponding generator matrix for E_8, showing that E_8 can be 3-integrated in 12 coordinates. Blank entries are zero.

Figure 2. Eutactic star of scale 3 in $E_7^{2+} \oplus \mathrm{I}_2$, containing $5 + 5 + 3 + 3 = 16$ vectors.

List of Table Captions

Table 1. Eutactic stars of scale 4 in the Niemeier lattices (the notation is described below Theorem 14).

Table 2. Construction of n-dimensional unimodular lattices Λ_n from Niemeier lattices L_{24}.

Table 1

components	star	number
D_{24}	$-$	24
$D_{16}E_8$	-0	16
	$0-$	8
E_8^3	(-00)	3×8
A_{24}	$+$	6×3
	5	6
D_{12}^2	(-0)	2×12
$A_{17}E_7$	$+0$	4×3
	$0-$	6
	$**$	2
	60	4
$D_{10}E_7^2$	-00	10
	$0(-0)$	2×6
	$0**$	2
$A_{15}D_9$	$+0$	4×3
	$0-$	9
	80	3
D_8^3	(-00)	3×8
A_{12}^2	$(+0)$	6×3
	15	3
	81	3
$A_{11}D_7E_6$	$+00$	3×3
	$0-0$	7
	$00-$	5
	401	3

Table 1 (cont.)

components	star	number
E_6^4	(-000)	4×5
	0111	1
	1(012)	3
$A_9^2 D_6$	$(+0)0$	4×3
	$00-$	6
	$**0$	2
	240	2
	480	2
D_6^4	-000	4×6
A_8^3	$(+00)$	6×3
	(411)	3×2
$A_7^2 D_5^2$	$(+0)00$	4×3
	$00(-0)$	2×5
	4400	2
A_6^4	$(+000)$	4×3
	$(*0*0)$	2×2
	5111	1
	1(216)	3
	0124	1
	1045	1
	2301	1
	4260	1

Table 1 (concluded)

components	star	number
$A_5^4 D_4$	(+000)0	4 × 3
	0000–	4
	(∗0∗0)0	2 × 2
	02220	1
	2(024)0	3
D_4^6	(–00000)	6 × 4
A_4^6	(+00000)	6 × 3
	011111	1
	1(01441)	5
A_3^8	(+0000000)	8 × 3
A_2^{12}	(∗00000) (∗00000)	6 × 2
	100000 011111	1
	0(10000) 2(01221)	5
	011111 100000	1
	2(02112) 0(10000)	5
A_1^{24}	$(*0^{11})\ (*0^{11})$	12 × 2
Λ_{24}	$8^{-1/2}(40^{11})(\pm 40^{11})$	12 × 2

(The last line does not follow the conventions used elsewhere in the table.)

Table 2

L_{24}	$n = 23$	$n = 22$	$n = 21$	20	19	18	17	16	15	14	12	8	0
D_{24}				$+^{b}$									$-^{a}$
$D_{16}E_8$	10	$**^{b}$		$+0^{b}$				$0-^{a}$				-0^{a}	
E_8^3		$**0^{b}$						-00^{a}					
A_{24}	5^{a}		$+^{a}$										
D_{12}^2	12	$**^{b}$		$+0^{b}$							-0^{a}		
$A_{17}E_7$	60^{a}, 31	$**^{b}$	$+0^{a}$			$0-^{a}$							
$D_{10}E_7^2$	110, 211	$0**^{a}$, $**0^{b}$		$+00^{b}$		$0-0^{a}$				-00^{a}			
$A_{15}D_9$	80^{a}, 21, 42	$**^{b}$	$+0^{a}$	$0+^{b}$					$0-^{a}$				
D_8^3	033, 122	$**0^{b}$		$+00^{b}$				-00^{a}					
A_{12}^2	15^{a}, 32	$**^{b}$	$+0^{a}$										
$A_{11}D_7E_6$	620, 401^{a}, 330, 111, 222	$0**^{b}$, $*0*^{b}$, $**0^{b}$	$+00^{a}$	$0+0^{b}$	$00-^{a}$		$0-0^{a}$						

Table 2 (concluded)

L_{24}	$n = 23$	$n = 22$	$n = 21$	20	19	18
E_6^4	0111^a	$\ast\ast 00^b$			-000^a	
$A_9^2 D_6$	$501, 240^a, 312, 121$	$\ast\ast 0^a, \ast 0 \ast^b$	$+00^a$	$00+^b$		$00-^a$
D_6^4	$2222, 0123$	$\ast\ast 00^b$		$+000^b$		-000^a
A_8^3	$036, 411^a, 177$	$\ast\ast 0^b$	$+00^a$			
$A_7^2 D_5^2$	$4400^a, 4022, 2031, 2220, 1112, 1303$	$00\ast\ast^b, \ast\ast 00^b, \ast 0 \ast 0^b$	$+000^a$	$00+0^b$	$00-0^a$	
A_6^4	$5111^a, 0124^a$	$\ast\ast 00^a$	$+000^a$			
$A_5^4 D_4$	$00331, 02220^a, 31110, 04111$	$\ast 000 \ast^b, \ast\ast 000^a$	$+0000^a$	$0000+^b$		
D_4^6	002332	$\ast\ast 0000^b$		$+0^{5\ b}$		
A_4^6	$011111^a, 001234$	$\ast\ast 0000^b$	$+0^{5\ a}$			
A_3^8	$00002222, 00021113$	$\ast\ast 000000^b$	$+0^{7\ a}$			
A_2^{12}	$0^6 1^{6\ a}$	$\ast\ast 0^{10\ a}$				
A_1^{24}	$0^{16} 1^8$	$\ast\ast 0^{22\ a}$				
Λ_{24}	minimal vector[a]					

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