

A Zador-Like Formula for Quantizers Based on Periodic Tilings

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ABSTRACT

We consider Zador's asymptotic formula for the distortion-rate function for a variable-rate vector quantizer in the high-rate case. This formula involves the differential entropy of the source, the rate of the quantizer in bits per sample, and a coefficient G which depends on the geometry of the quantizer but is independent of the source. We give an explicit formula for G in the case when the quantizing regions form a periodic tiling of n -dimensional space, in terms of the volumes and second moments of the Voronoi cells. As an application we show, extending earlier work of Kashyap and Neuhoff, that even a variable-rate three-dimensional quantizer based on the "A15" structure is still inferior to a quantizer based on the body-centered cubic lattice. We also determine the smallest covering radius of such a structure.

1. Introduction

Zador gave two asymptotic formulae for the distortion-rate function for a vector quantizer in \mathbb{R}^n , depending on whether a fixed- or variable-rate code is used to transmit which cell the sample point belongs to. These are treated as cases (A) and (B) in [15]. See also Equations (19) and (20) in [7]. In the present note we are concerned with the variable-rate case. Zador's formula for this case states that the distortion-rate function has the form

$$\delta(R) \cong G 2^{2h(X)} 2^{-2R}, \quad (1)$$

where $\delta(R)$ is the average squared error per symbol, $h(X)$ is the differential entropy per dimension of the source X , R bits/symbol is the rate of the quantizer, and G depends on the quantizer but not on the source. This is from page 4 of [15] and Equation (20) in [7]. Note that, as pointed out in [7], the version of Eq. (1) given in [16] is incorrect. For further information see also [14] and [6],

Chapter 5. The value of G can therefore be used to compare different quantizers. To calculate G we may choose any convenient distribution for the source X .

Suppose first that the quantizer points form a lattice $\Lambda \subseteq \mathbb{R}^n$, with Voronoi cell \mathcal{V} of volume $V = \sqrt{\det \Lambda}$. To calculate G we assume that X is uniformly distributed over \mathcal{V} . Then G becomes

$$G = \frac{1}{n} \frac{\int_{\mathcal{V}} \|x\|^2 dx}{V^{1+2/n}}, \quad (2)$$

the familiar expression for the average mean squared error per dimension of a lattice quantizer ([3], [4, Chap. 2], [5, Chap. 10], [6, Chap. 5]). In this case all the quantizing cells have the same volume, and the fixed-rate and variable-rate versions of Zador's formula coincide.

The main purpose of this note is to put on record an analogue of (2) (see Eq. (8) and (9)) for the variable-rate Zador formula which applies to the case when the quantizing regions form a periodic tiling of \mathbb{R}^n , such as that formed by the Voronoi cells for a union of a finite number of translates of a lattice. This question arose in a recent investigation of quantizers that are based on writing a lattice as an intersection of several simpler lattices [11].

A second motivation for our work was a recent paper of Kashyap and Neuhoff [8], which makes use of a fixed-rate analogue of (2) for periodic lattice quantizers (see Eq. (10) and (14) below). One of the goals of [8] was to see if the A15 arrangement of points in \mathbb{R}^3 that has recently arisen in several different contexts ([9], [10], [12], [13]) could produce a better three-dimensional quantizer than the body-centered cubic (bcc) lattice. The latter is known to be the best *lattice* quantizer in \mathbb{R}^3 [1], but the question of the existence of a better nonlattice quantizer remains open.

Kashyap and Neuhoff [8] found that with their figure of merit the best quantizer based on the A15 arrangement is inferior to the bcc lattice. In Section 3 we repeat the comparison using our figure of merit. Now a different version of the A15 quantizer is best, but is still inferior to the bcc lattice.

Other applications of our formula will be found in [11].

Another open problem in three-dimensional geometry is to determine the best covering of \mathbb{R}^3 by equal (overlapping) spheres. In Section 3 we determine the smallest covering radius that can be achieved with the A15 structure. This is also (slightly) worse than that of the bcc lattice.

2. Periodic quantizers

Consider a vector quantizer in \mathbb{R}^n in which the quantizing regions form a periodic tiling. Let \mathcal{V} be a minimal periodic unit or tile for the tiling, and let $\mathcal{P}_1, \dots, \mathcal{P}_k$ be a list of the different polytopes

occurring among the quantizing regions. Let c_i be the centroid of \mathcal{P}_i . In order to determine G we assume the source X is uniformly distributed over \mathcal{V} , and let p_i ($i = 1, \dots, k$) be the probability that X is in a cell of type \mathcal{P}_i . Also let $N_i = p_i V / V_i$, where $V_i = \text{vol } \mathcal{P}_i$ and $V = \text{vol } \mathcal{V}$. Then N_i is the number of cells of type \mathcal{P}_i per copy of \mathcal{V} , and

$$V = N_1 V_1 + \dots + N_k V_k . \quad (3)$$

For example, let $\Lambda \subseteq \mathbb{R}^n$ be a lattice and let $a_1 + \Lambda, \dots, a_r + \Lambda$ ($a_i \in \mathbb{R}^n$) be distinct translates. Then the Voronoi cells for the union of the points $a_i + \Lambda$ ($i = 1, \dots, r$) form a quantizer of the type considered here.

We now apply (1). The left-hand side is the normalized mean squared error per dimension $U/(nV)$, where

$$U = \sum_{i=1}^k N_i U_i = V \sum_{i=1}^k \frac{p_i U_i}{V_i} \quad (4)$$

and

$$U_i = \int_{\mathcal{P}_i} \|x - c_i\|^2 dx \quad (5)$$

is the unnormalized mean squared error over a cell of type \mathcal{P}_i .

The differential entropy per dimension is

$$h(X) = \frac{1}{n} \log_2 V, \quad \text{so} \quad 2^{2h(X)} = V^{2/n} . \quad (6)$$

It remains to calculate the rate R of the quantizer. Observe that we need $H(p_1, \dots, p_k) = -\sum_{i=1}^k p_i \log_2 p_i$ bits to specify the type of cell to which the quantized point belongs, and a further $\sum_{i=1}^k p_i \log_2 N_i = \sum_{i=1}^k p_i \log_2 (p_i V / V_i)$ bits to specify the particular one of the N_i cells of that type. This requires a total of $\log_2 V - \sum_{i=1}^k p_i \log_2 V_i$ bits, and then R is this quantity divided by n , so that

$$2^{-2R} = V^{-2/n} \prod_{i=1}^k V_i^{2p_i/n} . \quad (7)$$

We substitute (4), (6), (7) into (1) to get our expression

$$G = \frac{\sum_{i=1}^k \frac{p_i U_i}{V_i}}{n \left(\prod_{i=1}^k V_i^{p_i} \right)^{\frac{2}{n}}} , \quad (8)$$

for the average mean squared error per dimension using variable-rate coding. The numerator of (8) is equal to U/V (see (4)), so we may rewrite (8) as

$$G = \frac{U}{nV \prod_{i=1}^k V_i^{2p_i/n}} \quad (9)$$

In contrast, the expression given by Kashyap and Neuhoff [8] for fixed-rate coding is the following. Suppose the basic tile \mathcal{V} contains L cells, Q_1, \dots, Q_L , not assumed to be distinct, where Q_i has volume V_i and unnormalized second moment U_i (as in (5)). Then their expression is

$$G = \frac{L^{2/n} \sum_{i=1}^L U_i}{nV^{1+2/n}}, \quad (10)$$

where $V = \text{vol } \mathcal{V}$. The following justification is equivalent to the one in [8], but clarifies the difference between our two approaches. Consider a large region of space, B , which is partitioned into λ copies of \mathcal{V} , and let X be uniformly distributed over B . The left-hand side of (1) is

$$\frac{\lambda \sum_{i=1}^L U_i}{n \text{vol } B} = \frac{\sum_{i=1}^L U_i}{nV}. \quad (11)$$

The differential entropy per dimension is

$$h(X) = \frac{1}{n} \log_2(\text{vol } B), \quad \text{so } 2^{2h(X)} = (\lambda V)^{2/n}. \quad (12)$$

To compute the rate, it may be argued that $\log_2 \lambda$ bits are required to specify the copy of \mathcal{V} , and $\log_2 L$ bits to specify which of the Q_i 's the point belongs to. This requires a total of $\log_2 \lambda L$ bits, so $R = (1/n) \log_2 \lambda L$, and

$$2^{-2R} = (\lambda L)^{-2/n}. \quad (13)$$

Arguing as before, we substitute (11)–(13) into (1), which gives (10).

If there are k different types of cells among the Q_i , with the j -th cell occurring N_j times, then (10) can be written as

$$G = \frac{\left(\sum_{i=1}^k N_i\right)^{2/n} U}{nV^{1+2/n}}. \quad (14)$$

The ratio of the two expressions, (9) divided by (14), can be written as

$$2^{-\frac{2}{n}(\log_2 L - \sum_{i=1}^k p_i \log_2(p_i/N_i))}. \quad (15)$$

Both formulae, (9) and (14), depend only on the geometry of the quantizer. The difference between the two expressions arises because, as long as the cells do not all have the same volume, variable-length coding can take advantage of the different cell probabilities to reduce the overall rate.

If all cells have the same volume then (9) and (14) coincide, and if there is only type of cell then they both reduce to (2).

In general, the fact that (9) is less than equal to (14) can be shown directly. After canceling some common factors and rearranging, we must show that

$$\prod_{j=1}^k V_j^{p_j} \geq \frac{1}{\sum_{i=1}^k \frac{p_i}{V_i}},$$

and this follows from the geometric-mean harmonic-mean inequality.

3. Quantizers based on the A15 structure

The A15 structure was discovered by Kasper et al. [9] in the clathrate compound $\text{Na}_8\text{Si}_{46}$. It was used by Weaire and Phelan ([13]; see also [12]) to construct a counter-example to Kelvin’s conjecture on minimal surface soap films. It was also used by Lagarias and Shor [10] to construct counter-examples to Keller’s conjecture in dimensions 10 and above. Since the soap film problem attempts to find a partition of space into cells which are good approximations to spheres, it was therefore natural to ask if the A15 structure could also lead to a record-breaking quantizer in three dimensions.

The A15 structure may be defined as the union of eight translates of the cubic lattice $4\mathbb{Z}^3$ by the vectors $(0, 0, 0)$, $(2, 2, 2)$, $(0 \pm 1, 2)$, $(2, 0, \pm 1)$, $(\pm 1, 2, 0)$. (The first two translates alone give the bcc lattice.)

There are two types of points in this structure, the *even* points in which all coordinates are even, and the *odd* points in which some coordinate is odd. There are isometries of \mathbb{R}^3 which map A15 to itself and act transitively on the even points and on the odd points. However, even points are not equivalent to odd points.

Weaire and Phelan consider a weighted Voronoi decomposition of \mathbb{R}^3 : walls between points of the same type occur along perpendicular bisectors, but a wall between an even point E and an odd point D is defined by the plane

$$(X - E) \cdot (D - E) = \mu(D - E) \cdot (D - E) ,$$

where μ is a weighting factor to be determined. For $0 < \mu < 3/5$ there are two types of Voronoi cells: 12-sided polyhedra \mathcal{P}_1 centered at the even points and 14-sided polyhedra \mathcal{P}_2 centered at the odd points. Weaire and Phelan use the “Surface Evolver” computer program of Brakke [2] to perturb the polyhedra so that they have equal volumes and minimal total surface area.

Kashyap and Neuhoff [8] use the weighted Voronoi decomposition based on the A15 structure, but adjust μ to give the smallest value of (10). The formula are simpler if we set $\mu = 2\alpha/5$. Then Kashyap and Neuhoff find (and we have confirmed) that

$$\begin{aligned} V_1 &= \text{vol } \mathcal{P}_1 = 4\alpha^3 , \\ V_2 &= \text{vol } \mathcal{P}_2 = \frac{4}{3}(8 - \alpha^3) , \end{aligned}$$

$$\begin{aligned}
U_1 &= \frac{71}{30}\alpha^5, \\
U_2 &= \frac{1}{90}(1200 - 600\alpha^3 + 360\alpha^4 - 71\alpha^5).
\end{aligned}$$

The fundamental tile \mathcal{V} has volume $V = 64$, and $L = 8$, $N_1 = 2$, $N_2 = 6$, $p_1 = \alpha^3/8$, $p_2 = 1 - \alpha^3/8$. Then (10) becomes

$$\frac{8^{2/3}(2U_1 + 6U_2)}{3 \cdot 64^{5/3}} = \frac{1}{96}(10 - 5\alpha^3 + 3\alpha^4). \quad (16)$$

This has a minimal value of 0.07873535... at $\alpha = 5/4$. For comparison, the values of G for the bcc and fcc (face-centered cubic) lattices are $19/(192 \cdot 2^{1/3}) = 0.07854328\dots$ and $2^{-11/3} = 0.07874507\dots$, respectively ([1], [3], [4]).

Using our formula (8) we find that

$$G = \frac{2U_1 + 6U_2}{192(V_1^{\alpha^3/8} V_2^{1-\alpha^3/8})^{2/3}} \quad (17)$$

which has a minimal value of 0.07872741... at $\alpha = 1.2401\dots$. This is slightly better, but still inferior to the bcc lattice.

Another unsolved problem is to find the best covering of \mathbb{R}^3 by overlapping spheres (cf. [4, Chap. 2, Table 2.1]). The bcc lattice is the best *lattice* covering, with thickness 1.4635..., but the question of the existence of a better nonlattice covering remains open. We find that the covering radius of the weighted Voronoi decomposition of A15 is minimized at $\alpha = 5/4$, which gives a thickness of

$$\frac{125\sqrt{3}\pi}{432} = 1.5745\dots,$$

just slightly worse than that of the bcc lattice (but again better than the fcc lattice, which has thickness 2.0944...).

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