An Alternative Converse Proof For the Gaussian Broadcast Channel Capacity

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***Abstract*—We provide an alternative converse proof for the well-known capacity formula of the Gaussian scalar broadcast channel. The original proof by Bergmans is a proof by contradiction, while the alternative proof is a direct proof.**

The capacity region for the additive white Gaussian broadcast channel under average power constraint S, with noise power $N_1 < N_2 < ... < N_K$, is given as (see [1])

$$R_i \leq \frac{1}{2}\log\left(1+\frac{\alpha_i S}{N_i+\sum_{j=1}^{i-1}\alpha_j S}\right), \quad i=1,2,...,K, \tag{1}$$

with $\alpha_i \geq 0$, and $\sum_{i=1}^{K}\alpha_i \leq 1$. Note that the rates are for the individual receivers without common messages.

Bergmans [1] gave a converse proof of this result by a contradiction argument; see [2] for the history on this result. In this short note, we give an alternative converse proof which is a direct proof.

Converse proof of Gaussian broadcast capacity:

Denote the independent messages for the receivers as $(W_1, W_2, ..., W_K)$, and consider the follows

$$\begin{aligned} nR_i &= H(W_i) = H(W_i|W_{i+1}^K) \\ &\leq [H(W_i|W_{i+1}^K) - H(W_i|\boldsymbol{Y_i}, W_{i+1}^K)] + \epsilon \\ &= I(W_i;\boldsymbol{Y_i}|W_{i+1}^K) + \epsilon \\ &= [h(\boldsymbol{Y_i}|W_{i+1}^K) - h(\boldsymbol{Y_i}|W_i^K)] + \epsilon \end{aligned} \tag{2}$$

where we have used Fano's inequality and $\epsilon \to 0$ as the decoding error $P_e \to 0$.

Define for $i = 1, 2, ..., K$

$$\hat{R}_i = \frac{1}{n}h(\boldsymbol{Y_i}|W_{i+1}^K) - \frac{1}{n}h(\boldsymbol{Y_i}|W_i^K)$$
$$r_i = \frac{1}{n}h(\boldsymbol{Y_i}|W_{i+1}^K).$$

Clearly we have

$$r_1 - \hat{R}_1 = \frac{1}{n}h(\boldsymbol{Y_1}|W_1^K) = \frac{1}{2}\log 2\pi e N_1. \tag{3}$$

It is also clear that

$$\begin{aligned} r_K &= \frac{1}{n}h(\boldsymbol{Y_K}) \leq \frac{1}{n}\sum_{k=1}^{n} h(Y_K(k)) \\ &\leq \sum_{k=1}^{n}\frac{1}{2n}\log(2\pi e)^n[\mathbb{E}(Y_K(k))^2] \\ &= \sum_{k=1}^{n}\frac{1}{2n}\log(2\pi e)^n[N_K + \mathbb{E}(X(k))^2] \\ &\leq \frac{1}{2}\log 2\pi e[N_K + S], \end{aligned} \tag{4}$$

where the last inequality follows from the concavity of the $\log(\cdot)$ function and the power constraint.

By the entropy power inequality [3], we have

$$\begin{aligned} r_i - \hat{R}_i &= \frac{1}{n}h(\boldsymbol{Y_i}|W_i^K) \\ &\geq \frac{1}{2}\log\left[\exp(\frac{2}{n}h(\boldsymbol{Y_{i-1}}|W_i^K)) + (2\pi e)(N_i - N_{i-1})\right] \\ &= \frac{1}{2}\log\left[\exp(2r_{i-1}) + (2\pi e)(N_i - N_{i-1})\right]. \end{aligned} \tag{5}$$

It follows that for $i = 1, 2, ..., K$

$$\exp(2r_i) \geq [\exp(2r_{i-1}) + (2\pi e)(N_i - N_{i-1})]\exp(2\hat{R}_i), \tag{6}$$

where for convenience we defined $r_0 \triangleq -\infty$ and $N_0 \triangleq 0$. From (3), (4) and (6), we have

$$\exp(2r_1) - (2\pi e)N_1 \geq 0 \tag{7}$$
$$\exp(2r_K) - (2\pi e)N_K \leq (2\pi e)S \tag{8}$$
$$\exp(2r_i) - (2\pi e)N_i \geq \exp(2r_{i-1}) - (2\pi e)N_{i-1}, \quad i = 2, 3, ..., K. \tag{9}$$

This implies that $\exp(2r_i) - (2\pi e)N_i$, $i = 1, 2, ..., K$, is a monotone sequence in the range of $[0, (2\pi e)S]$, and we can now write

$$\exp(2r_i) = (2\pi e)N_i + (2\pi e)\sum_{j=1}^{i}\alpha_j S, \quad i = 1, 2, ..., K, \tag{10}$$

for some $\alpha_i \geq 0$, and $\sum_{i=1}^{K}\alpha_i \leq 1$. At this point it is rather clear that from (6) we have

$$0 \leq R_i - \epsilon \leq \hat{R}_i \leq \frac{1}{2}\log\left(1 + \frac{\alpha_i S}{N_i + \sum_{j=1}^{i-1}\alpha_j S}\right), \quad i = 1, 2, ..., K, \tag{11}$$

for some $\alpha_i \geq 0$, $i = 1, 2, ..., K$ and $\sum_{j=1}^{K}\alpha_j \leq 1$. Since ϵ is arbitrarily small, the closure of this region is exactly the one given in (1). ■

Remark: Unfortunately the proof given here appears difficult to generalize to the multiple-input multiple-output (MIMO) Gaussian broadcast channel case, for which Weingarten *et al.* provided a converse proof by generalizing Bergmans' proof together with the enhancing technique [4].

References

[1] P. Bergmans, "A simple converse proof for the broadcast channels with additive white Gaussian noise," *IEEE Trans. Information Theory*, vol. 20, no. 2, pp. 279–280, Mar. 1974.

[2] T. Cover, "Comments on broadcast channels," *IEEE Trans. Information Theory*, vol. 44, no. 6, pp. 2524–2530, Oct. 1998.

[3] T. M. Cover and J. A. Thomas, *Elements of information theory*, New York: Wiley, 1991.

[4] H. Weingarten, Y. Steinberg, and S. Shamai, "The capacity region of the Gaussian multiple-input multiple-output broadcast channel," *IEEE Trans. Information Theory*, vol. 52, no. 9, pp. 3936–3964, Sep. 2006.